Complementarity Models for Rolling and Plasticity

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1. Time-Stepping Schemes and Their Relaxation

Nonsmooth contact dynamics—what is it?

- Differential problem with variational inequality constraints – DVI

Newton Equations

Non-Penetration Constraints

$$M\frac{dv}{dt} = \sum_{j=1,2,\ldots,p}\left(c_n^{(j)}n^{(j)} + \beta_1^{(j)}t_1^{(j)} + \beta_2^{(j)}t_2^{(j)}\right) + f_c(q,v) + k(t,q,v)$$

$$\frac{dq}{dt} = \Gamma(q)v$$

Generalized Velocities

$$c_n^{(j)} \geq 0 \quad \perp \quad \Phi^{(j)}(q) \geq 0, \qquad j = 1,2,\ldots,p$$

$$\left(\beta_1^{(j)}, \beta_2^{(j)}\right) = \operatorname*{argmin}_{\mu^{(j)}c_n^{(j)} \geq \sqrt{\left(\beta_1^{(j)}+\beta_2^{(j)}\right)^2}} \left[\left(v^T t_1^{(j)}\right)\beta_1 + \left(v^T t_2^{(j)}\right)\beta_2\right]$$

Friction Model

- Truly, a Differential Problem with Equilibrium Constraints

Time stepping scheme -- original

- A measure differential inclusion solution can be obtained by time-stepping (Stewart, 1998, Anitescu 2006)

$$M\left(\boldsymbol{v}^{(l+1)} - \boldsymbol{v}^{l}\right) = \sum_{i \in \mathcal{A}(q^{(l)},\epsilon)} \left(\gamma_n^i \boldsymbol{D}_n^i + \gamma_u^i \boldsymbol{D}_u^i + \gamma_v^i \boldsymbol{D}_v^i\right) + \sum_{i \in \mathcal{G}_\mathcal{B}} \left(\gamma_b^i \nabla \Psi^i\right) + h \boldsymbol{f}_t\left(t^{(l)}, \boldsymbol{q}^{(l)}, \boldsymbol{v}^{(l)}\right)$$

$$0 = \frac{1}{h}\Psi^i\left(\boldsymbol{q}^{(l)}\right) + \nabla \Psi^{i^T} \boldsymbol{v}^{(l+1)} + \frac{\partial \Psi^i}{\partial t}, \quad i \in \mathcal{G}_\mathcal{B}$$

$$0 \leq \frac{1}{h}\Phi^i\left(\boldsymbol{q}^{(l)}\right) + \nabla \Phi^{i^T} \boldsymbol{v}^{(l+1)} \quad \perp \quad \gamma_n^i \geq 0, \; i \in \mathcal{A}(q^{(l)},\epsilon)$$

$$\left(\gamma_u^i, \gamma_v^i\right) = \operatorname{argmin}_{\mu^i \gamma_n^i \geq \sqrt{(\gamma_u^i)^2 + (\gamma_v^i)^2}} \left[\boldsymbol{v}^T\left(\gamma_u \boldsymbol{D}_u^i + \gamma_v \boldsymbol{D}_v^i\right)\right] \quad i \in \mathcal{A}(q^{(l)},\epsilon)$$

$$\boldsymbol{q}^{(l+1)} = \boldsymbol{q}^{(l)} + h \boldsymbol{v}^{(l+1)},$$

Speeds

Reaction impulses

Forces

Constraint Stabilization

Bilateral constraint equations

Contact constraint equations

COMPLEMENTARITY!

Coulomb 3D friction model

Pause: Constraint Stabilization

- Compared to original scheme

$$\nabla\Phi(q^{(l)})^T v^{(l+1)} \geq 0 \Longrightarrow \Phi^{(j)}(q^{(l)}) + \gamma h_l \nabla\Phi(q^{(l)})^T v^{(l+1)} \geq 0.$$

$$\nabla\Theta(q^{(l)})^T v^{(l+1)} = 0 \Longrightarrow \Theta^{(j)}(q^{(l)}) + \gamma h_l \nabla\Theta(q^{(l)})^T v^{(l+1)} = 0.$$

- Allows fixed time steps for plastic collisions.
- How do we know it is achieved? Infeasibility is one order better than accuracy (O(h^2))

Time Stepping -- Convex Relaxation

- A modification (relaxation, to get convex QP with conic constraints):

$$M(\boldsymbol{v}^{(l+1)} - \boldsymbol{v}^{l}) = \sum_{i \in \mathcal{A}(q^{(l)},\epsilon)} \left(\gamma_n^i \boldsymbol{D}_n^i + \gamma_u^i \boldsymbol{D}_u^i + \gamma_v^i \boldsymbol{D}_v^i\right) +$$

$$+ \sum_{i \in \mathcal{G}_{\mathcal{B}}} \left(\gamma_b^i \nabla \Psi^i\right) + h \boldsymbol{f}_t(t^{(l)}, \boldsymbol{q}^{(l)}, \boldsymbol{v}^{(l)})$$

$$0 = \frac{1}{h}\Psi^i(\boldsymbol{q}^{(l)}) + \nabla {\Psi^i}^T \boldsymbol{v}^{(l+1)} + \frac{\partial \Psi^i}{\partial t}, \quad i \in \mathcal{G}_{\mathcal{B}}$$

$$0 \leq \frac{1}{h}\Phi^i(\boldsymbol{q}^{(l)}) + \nabla {\Phi^i}^T \boldsymbol{v}^{(l+1)} \; - \mu^i \sqrt{(\boldsymbol{D}_u^{i,T} \boldsymbol{v})^2 + (\boldsymbol{D}_v^{i,T} \boldsymbol{v})^2}$$

$$\perp \quad \gamma_n^i \geq 0, \; i \in \mathcal{A}(q^{(l)}, \epsilon)$$

$$(\gamma_u^i, \gamma_v^i) = \mathrm{argmin}_{\mu^i \gamma_n^i \geq \sqrt{(\gamma_u^i)^2 + (\gamma_v^i)^2}} \quad i \in \mathcal{A}(q^{(l)}, \epsilon)$$

$$\left[\boldsymbol{v}^T (\gamma_u \boldsymbol{D}_u^i + \gamma_v \boldsymbol{D}_v^i)\right]$$

$$\boldsymbol{q}^{(l+1)} = \boldsymbol{q}^{(l)} + h \boldsymbol{v}^{(l+1)},$$

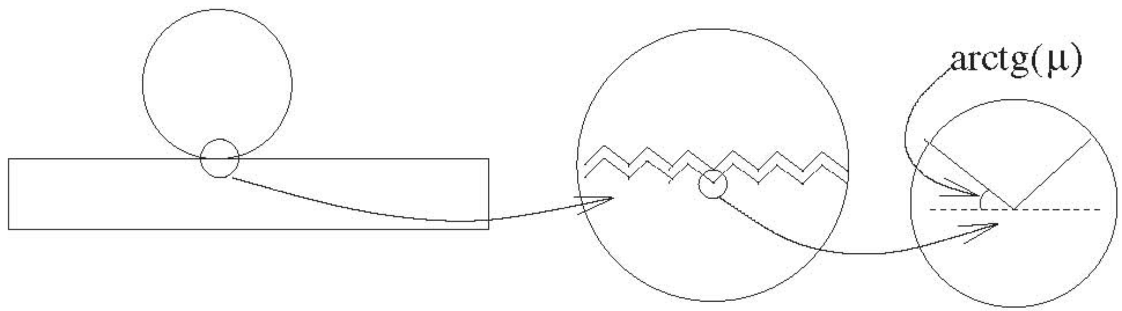
(For small m and/or small speeds, almost no one-step differences from the Coulomb theory)

But In any case, converges to same MDI as unrelaxed scheme.

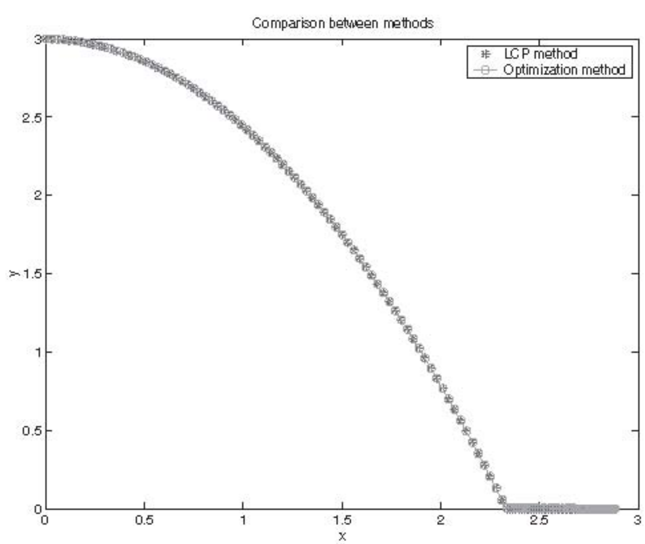
[see M.Anitescu, “Optimization Based Simulation of Nonsmooth Rigid Body Dynamics”]

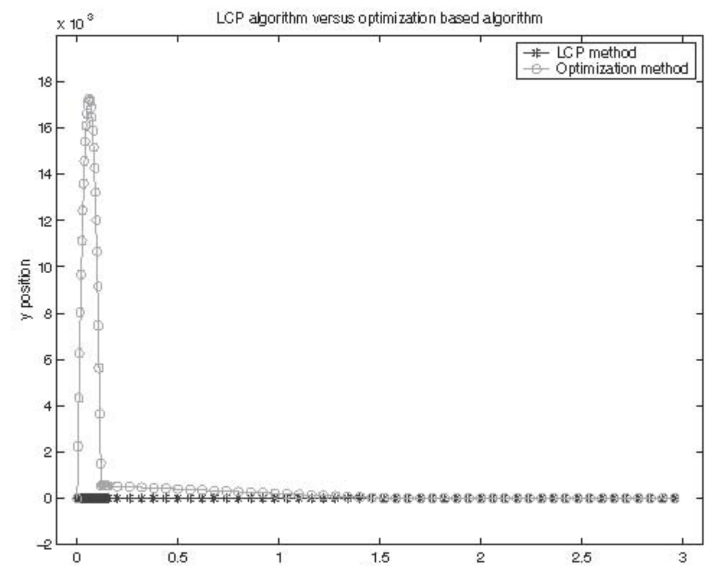
What is physical meaning of the relaxation?

- Origin



- Behavior





Further insight.

- The key is the combination between relaxation and constraint stabilization.

$$0 \leq \frac{1}{h}\Phi^{(j)}\left(q^{(l)}\right)+\nabla_q \Phi^{(j)}\left(q^{(l)}\right)v^{(l+1)}-\mu^{(j)}\sqrt{\left(D_u^{l,t}v\right)^2+\left(D_v^{l,t}v\right)^2}$$

- If the time step is smaller than the variation in velocity then the gap function settles at

$$0 \approx \frac{1}{h}\Phi^{(j)}\left(q^{(l)}\right)-\mu^{(j)}\sqrt{\left(D_u^{l,t}v\right)^2+\left(D_v^{l,t}v\right)^2}$$

- So the solution is the same as the original scheme for a slightly perturbed gap function…..

2. Cone complementarity time stepping

Cone complementarity

- Aiming at a more compact formulation:

$$\boldsymbol{b}_{\mathcal{A}} = \left\{ \frac{1}{h}\Phi^{i_1}, 0, 0, \frac{1}{h}\Phi^{i_2}, 0, 0, \dots, \frac{1}{h}\Phi^{i_{n_{\mathcal{A}}}}, 0, 0 \right\}$$

$$\boldsymbol{\gamma}_{\mathcal{A}} = \left\{ \gamma_n^{i_1}, \gamma_u^{i_1}, \gamma_v^{i_1}, \gamma_n^{i_2}, \gamma_u^{i_2}, \gamma_v^{i_2}, \dots, \gamma_n^{i_{n_{\mathcal{A}}}}, \gamma_u^{i_{n_{\mathcal{A}}}}, \gamma_v^{i_{n_{\mathcal{A}}}} \right\}$$

$$\boldsymbol{b}_{\mathcal{B}} = \left\{ \frac{1}{h}\Psi^1 + \frac{\partial \Psi^1}{\partial t}, \frac{1}{h}\Psi^2 + \frac{\partial \Psi^2}{\partial t}, \dots, \frac{1}{h}\Psi^{n_{\mathcal{B}}} + \frac{\partial \Psi^{n_{\mathcal{B}}}}{\partial t} \right\}$$

$$\boldsymbol{\gamma}_{\mathcal{B}} = \left\{ \gamma_b^1, \gamma_b^2, \dots, \gamma_b^{n_{\mathcal{B}}} \right\}$$

$$D_{\mathcal{A}} = \left[D^{i_1} | D^{i_2} | \dots | D^{i_{n_{\mathcal{A}}}} \right], \quad i \in \mathcal{A}(\boldsymbol{q}^l, \epsilon) \qquad D^i = [\boldsymbol{D}_n^i | \boldsymbol{D}_u^i | \boldsymbol{D}_v^i]$$

- $D_{\mathcal{B}} = \left[\nabla \Psi^{i_1} | \nabla \Psi^{i_2} | \dots | \nabla \Psi^{i_{n_{\mathcal{B}}}} \right], \quad i \in \mathcal{G}_{\mathcal{B}}$

$$\boldsymbol{b}_{\mathcal{E}} \in \mathbb{R}^{n_{\mathcal{E}}} = \{ \boldsymbol{b}_{\mathcal{A}}, \boldsymbol{b}_{\mathcal{B}} \}$$

$$\boldsymbol{\gamma}_{\mathcal{E}} \in \mathbb{R}^{n_{\mathcal{E}}} = \{ \boldsymbol{\gamma}_{\mathcal{A}}, \boldsymbol{\gamma}_{\mathcal{B}} \}$$

$$D_{\mathcal{E}} = [D_{\mathcal{A}} | D_{\mathcal{B}}]$$

Cone complementarity

- Also define:

$$\tilde{\boldsymbol{k}}^{(l)} = M\boldsymbol{v}^{(l)} + h\boldsymbol{f}_t(t^{(l)}, \boldsymbol{q}^{(l)}, \boldsymbol{v}^{(l)})$$

$$N = D_{\mathcal{E}}^T M^{-1} D_{\mathcal{E}}$$

$$\boldsymbol{r} = D_{\mathcal{E}}^T M^{-1} \tilde{\boldsymbol{k}} + \boldsymbol{b}_{\mathcal{E}}$$

- Then:

$$\begin{aligned}
M(\boldsymbol{v}^{(l+1)} - \boldsymbol{v}^{l}) &= \sum_{i \in \mathcal{A}(q^{(l)}, \epsilon)} \left(\gamma_n^i \boldsymbol{D}_n^i + \gamma_u^i \boldsymbol{D}_u^i + \gamma_v^i \boldsymbol{D}_v^i\right) + \\
&\quad + \sum_{i \in \mathcal{G}_{\mathcal{B}}} \left(\gamma_b^i \nabla \Psi^i\right) + h\boldsymbol{f}_t(t^{(l)}, \boldsymbol{q}^{(l)}, \boldsymbol{v}^{(l)}) \\
0 &= \frac{1}{h}\Psi^i(\boldsymbol{q}^{(l)}) + \nabla {\Psi^i}^T \boldsymbol{v}^{(l+1)} + \frac{\partial \Psi^i}{\partial t}, \quad i \in \mathcal{G}_{\mathcal{B}} \\
0 &\leq \frac{1}{h}\Phi^i(\boldsymbol{q}^{(l)}) + \nabla {\Phi^i}^T \boldsymbol{v}^{(l+1)} \\
&\quad \perp \quad \gamma_n^i \geq 0, \; i \in \mathcal{A}(q^{(l)}, \epsilon) \\
(\gamma_u^i, \gamma_v^i) &= \operatorname{argmin}_{\mu^i \gamma_n^i \geq \sqrt{(\gamma_u^i)^2 + (\gamma_v^i)^2}} \quad i \in \mathcal{A}(q^{(l)}, \epsilon) \\
&\quad \left[\boldsymbol{v}^T(\gamma_u \boldsymbol{D}_u^i + \gamma_v \boldsymbol{D}_v^i)\right] \\
\boldsymbol{q}^{(l+1)} &= \boldsymbol{q}^{(l)} + h\boldsymbol{v}^{(l+1)},
\end{aligned}$$

becomes..

$$(N\gamma_{\mathcal{E}} + \boldsymbol{r}) \in -\Upsilon^{\circ} \quad \perp \quad \gamma_{\mathcal{E}} \in \Upsilon$$

This is a CCP,

CONE COMPLEMENTARITY PROBLEM

Iterative methods for Cone Complementarity Problems

- The resulting cone complementarity problem

$$(N\boldsymbol{\gamma}_{\mathcal{E}} + \boldsymbol{r}) \in -\Upsilon^{\circ} \quad \perp \quad \boldsymbol{\gamma}_{\mathcal{E}} \in \Upsilon$$

- Our method: use a fixed-point iteration

$$\gamma^{r+1} = \lambda \Pi_{\Upsilon}\left(\gamma^{r} - \omega B^{r}\left(N\gamma^{r} + \boldsymbol{r} + K^{r}\left(\gamma^{r+1} - \gamma^{r}\right)\right)\right) + (1-\lambda)\,\gamma^{r}$$

- with matrices:
- ..and a non-extensive orthogonal projection operator onto feasible set

$$B^{r} = \begin{bmatrix} \eta_1 I_{n_1} & 0 & \cdots & 0 \\ 0 & \eta_2 I_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{n_k} I_{n_{n_k}} \end{bmatrix}$$

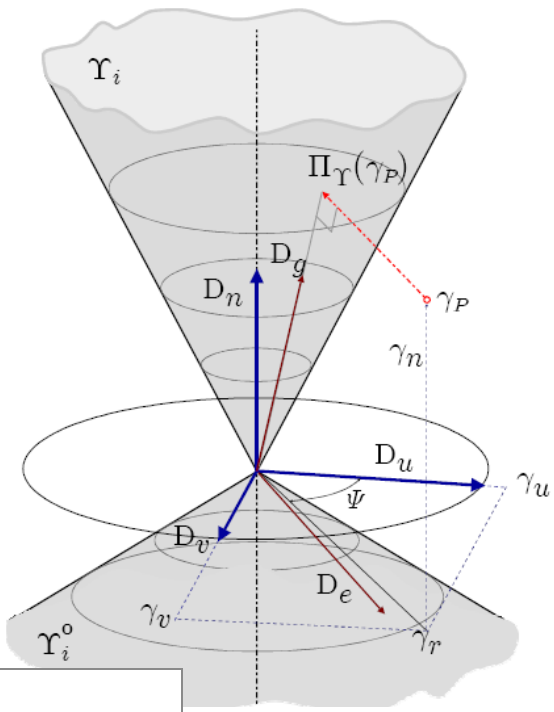
$$N^{T} = \begin{bmatrix} 0 & K_{12} & K_{13} & \cdots & K_{1n_k} \\ 0 & 0 & K_{23} & \cdots & K_{2n_k} \\ 0 & 0 & 0 & \cdots & K_{3n_k} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & & 0 \end{bmatrix}$$

$$\Pi_{\Upsilon} : \mathbb{R}^{n_{\mathcal{E}}} \rightarrow \mathbb{R}^{n_{\mathcal{E}}}$$

The projection operator is easy and separable

- *For each frictional contact constraint:*

$$\Pi_\Upsilon = \left\{ \Pi_{\Upsilon_1}(\boldsymbol{\gamma}_1)^T, \dots \Pi_{\Upsilon_{n_\mathcal{A}}}(\boldsymbol{\gamma}^{n_\mathcal{A}})^T, \Pi_b^1(\gamma_b^1), \dots, \Pi_b^{n_\mathcal{B}}(\gamma_b^{n_\mathcal{B}}) \right\}^T$$



- *For each bilateral constraint, simply do nothing.*
- The complete operator:

$$\forall i \in \mathcal{A}(\boldsymbol{q}^{(l)}, \epsilon)$$

$$\begin{array}{ll}
\gamma_r < \mu_i \gamma_n & \Pi_i = \boldsymbol{\gamma}_i \\
\gamma_r < -\dfrac{1}{\mu_i}\gamma_n & \Pi_i = \{0,0,0\} \\
\gamma_r > \mu_i \gamma_n \wedge \gamma_r > -\dfrac{1}{\mu_i}\gamma_n & \Pi_{i,n} = \dfrac{\gamma_r \mu_i + \gamma_n}{\mu_i^2 + 1} \\
 & \Pi_{i,u} = \gamma_u \dfrac{\mu_i \Pi_{i,n}}{\gamma_r} \\
 & \Pi_{i,v} = \gamma_v \dfrac{\mu_i \Pi_{i,n}}{\gamma_r}
\end{array}$$

Finding other Interesting Behaviors by Modeling the Force Constraint Set

- For computational reasons, we like the relaxation: it leads to solving convex optimization at each step.
- It seems to be easy to solve any time I can have a force constraint set that I can project easily onto. This implicitly (by duality) constrains the motion.
- Note that the set needs not be convex or even a cone.
- Are there other force/impulse constraint sets which model interesting behavior?

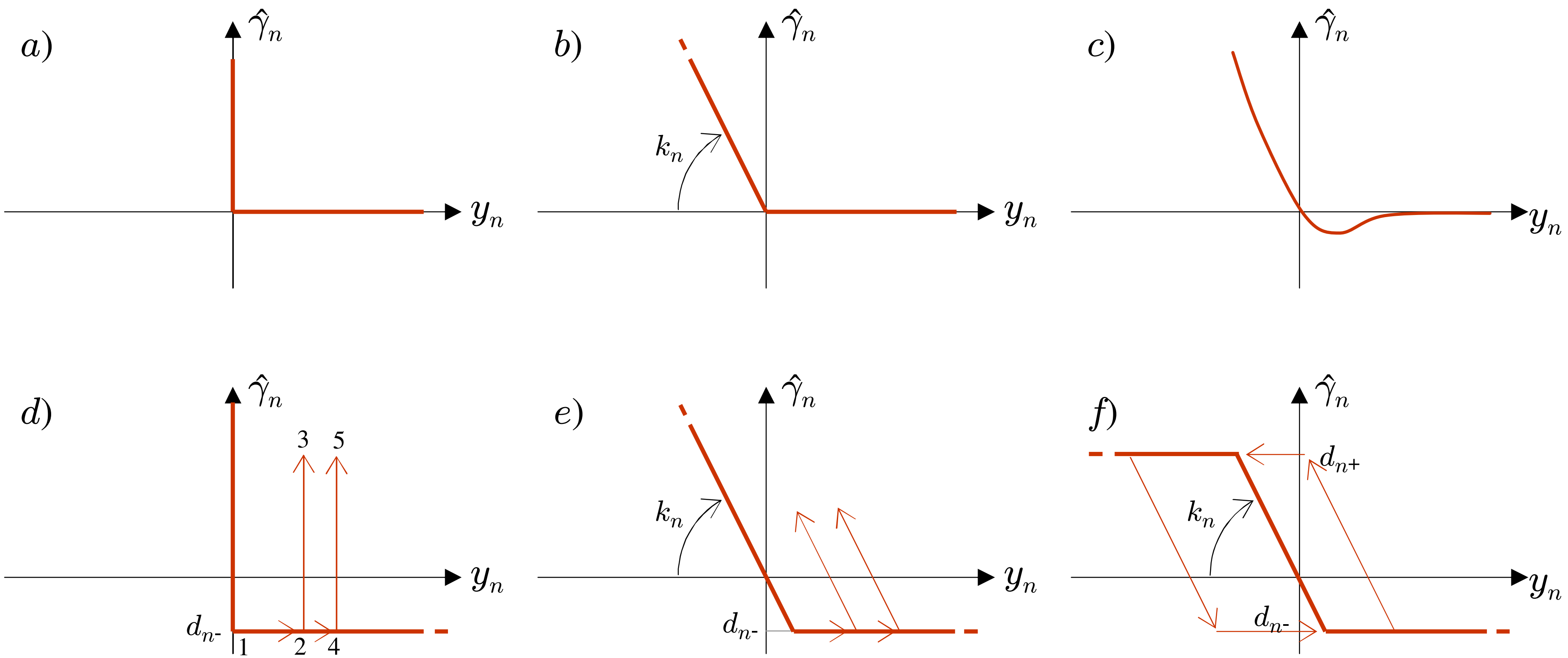
- So this is a "I have a hammer, where's the nail approach" : are there other cones that I may want to model that can lead to behaviors I would like to include?

- Of course there are several behaviors for which there are good guesses or even physics to this end.

3. Plasticity Models

Type of plasticity models in 1D.

- Plasticity elicits a different constitutive law compared to rigid contact



- Force-displacement relationship at a contact a) Rigid b) Compliant c) Nonlinear d) cohesion e) Cohesion + Compliance f) Cohesion + Compliance + plastic

The yield surface.

- Standard rigid contact 1-a can be turned to cohesive one by the transformation:

$$\hat{\gamma}_n \geq 0 \quad \Rightarrow \hat{\gamma}_n \in \left[d_n, \infty\right] = \Upsilon$$

- In addition, rule in 1-d satisfies (at contact):

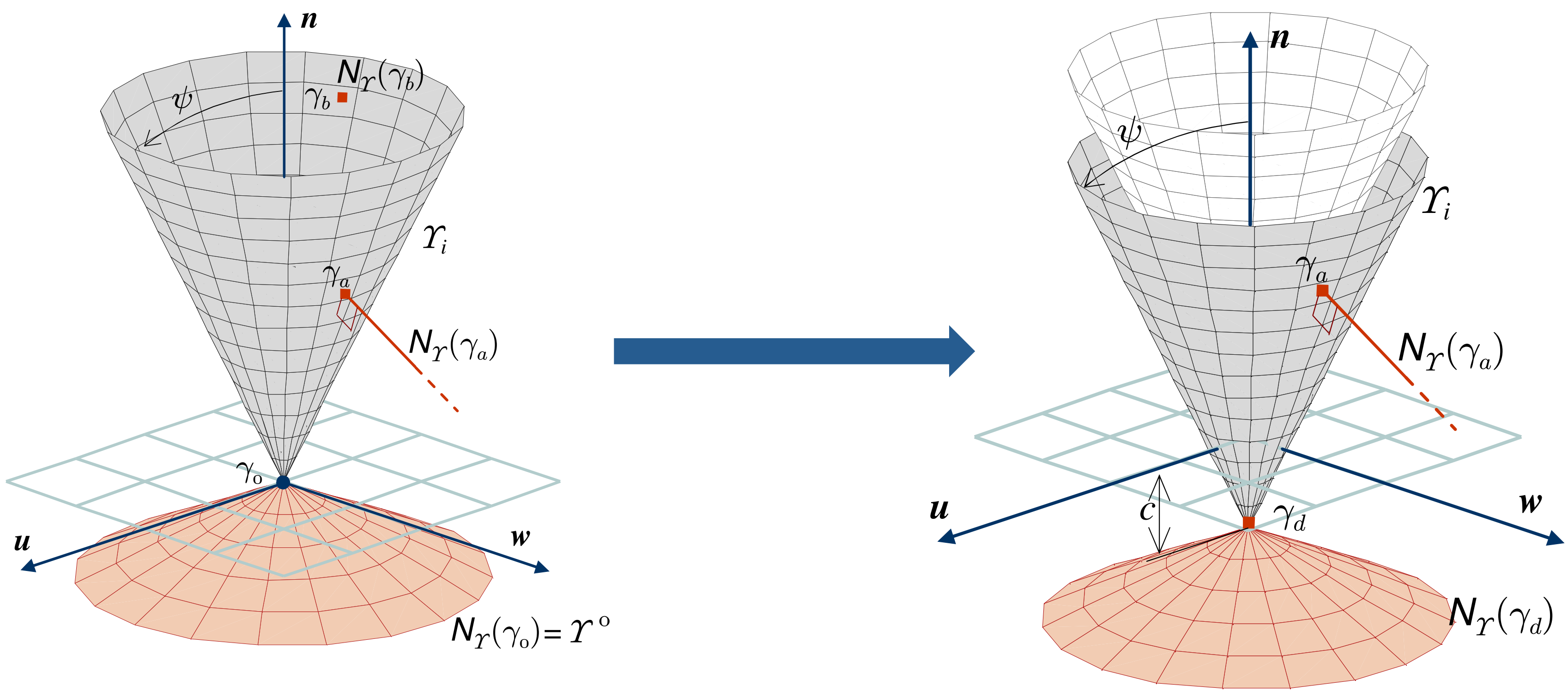
$$\Delta y_n \in -\mathcal{N}_{\Upsilon}\left(\hat{\gamma}_n\right)$$

- In this context Υ is called the yield surface.

- The displacement is normal at the Yield surface, such constitutive rules are called associative.

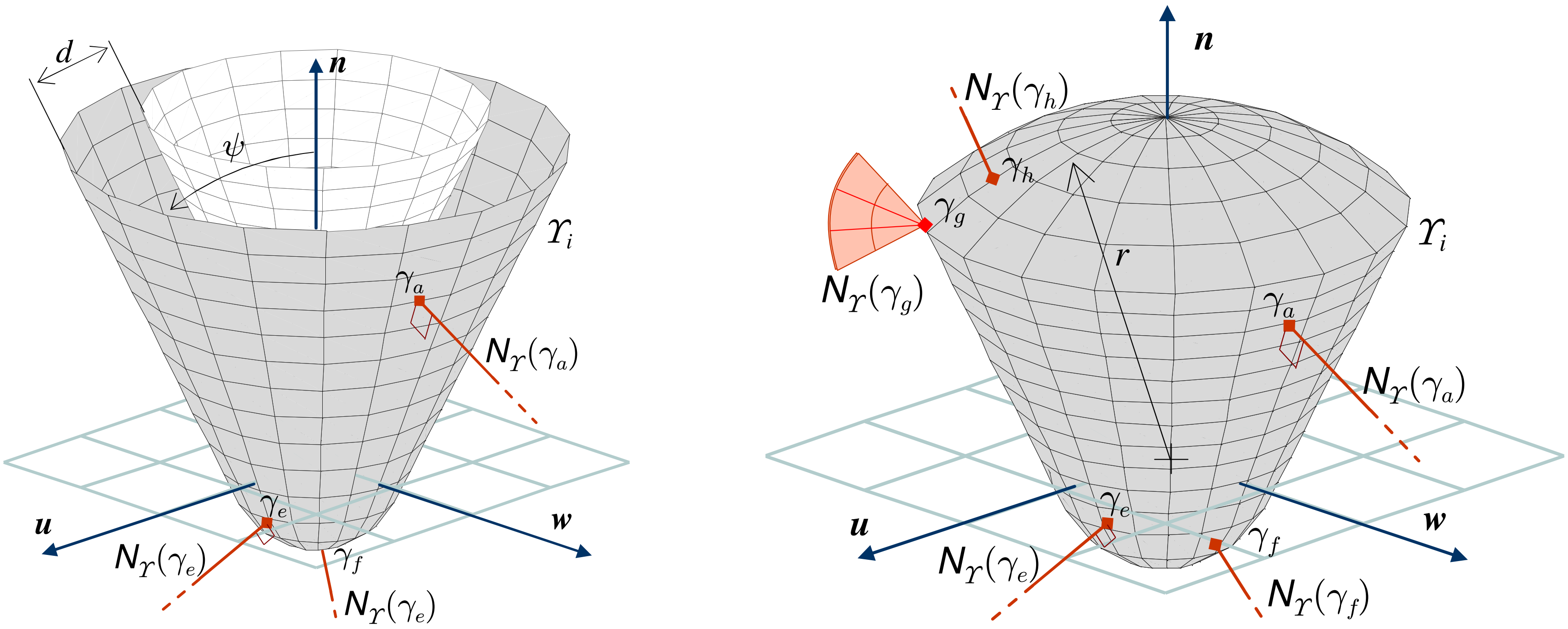
Example: Friction and Cohesion in 3D by playing around with Yield surfaces

- Shift the Coulomb cone downward and make it an associative yield surface.



Modeling plasticity

- The yield surface give by Coulomb needs to be modified as no infinite reaction is now allowed (crushing)



The model

- Separate elastic and plastic displacement:

$$\mathbf{y}^i = \mathbf{y}_E^i + \mathbf{y}_P^i$$

- The elastic part of the force allows to compute the force at the contact if the plastic part of the displacement is known.

$$\widehat{\gamma}_A^i = -K^i \left(\mathbf{y}^i - \mathbf{y}_P^i\right)$$

- Except, of course, the plastic part is NOT known***. But now we use the associated plasticity hypothesis to constrain the plastic displacement evolution with a variational inequality.***

$$\dot{\mathbf{y}}_P^i \in -\mathcal{N}_{\widehat{\Upsilon}^i}(\widehat{\gamma}_A^i) \quad ; \quad \widehat{\gamma}_A^i \in \widehat{\Upsilon}^i$$

- Mathematically, it is the same idea with normal velocity at the contact; EXCEPT that the containing set is no longer a cone, proper (it can be a shifted cone, or just some convex set. So we can time-step and close it.

Time-stepping

- After some notation

$$\dot{\mathbf{y}}_P^i = D^{i^T}\mathbf{v}^{l+1} + \left(h^2K^i\right)^{-1}\gamma_{\mathcal{A}}^{i,l+1} - \frac{1}{h}\left(\mathbf{y}^{i,l} - \mathbf{y}_P^{i,l}\right) \in -\mathcal{N}_{\Upsilon^i}(\gamma_{\mathcal{A}}^i)$$

- After this is solved, all we have to do is to solve the plasticity displacement

$$\mathbf{y}_P^{i,l+1} = \mathbf{y}_P^{i,l} + h\dot{\mathbf{y}}_P^i$$

- Use the velocity update

$$\mathbf{v}^{l+1} = \mathbf{v}^l + M^{-1}\sum_{i\in\mathcal{G}_{\mathcal{A}}} D^i\gamma_{\mathcal{A}}^{i,l+1} + hM^{-1}\mathbf{f}(\mathbf{q},\mathbf{v},t)$$

- Replace in the plasticity velocity equation; to obtain the variational inequality

$$\dot{\mathbf{y}}_P = D_{\mathcal{E}}{}^T M D_{\mathcal{E}}\gamma_{\mathcal{E}}^{l+1} + D_{\mathcal{E}}{}^T\left(\mathbf{v}^l + hM^{-1}\mathbf{f}(\mathbf{q},\mathbf{v},t)\right) - E_{\mathcal{E}}\gamma_{\mathcal{E}}^{l+1} - \mathbf{c} \in -\mathcal{N}_{\Upsilon}(\widehat{\gamma}_{\mathcal{E}})$$

- Over the cartesian product of cones: $\Upsilon = \underset{i\in\mathcal{G}_{\mathcal{A}}}{\times}\Upsilon^i$

Damping and VI

- We can similarly accommodate damping

$$\widehat{\gamma}_{\mathcal{A}}^{i} = -K^{i}\left(\mathbf{y}^{i} - \mathbf{y}_{P}^{i}\right) - R^{i}\left(\dot{\mathbf{y}}^{i} - \dot{\mathbf{y}}_{P}^{i}\right)$$

$$\dot{\mathbf{y}}_{P}^{i} \in -\mathcal{N}_{\widehat{\Upsilon}^{i}}(\widehat{\gamma}_{\mathcal{A}}^{i}) \quad ; \quad \widehat{\gamma}_{\mathcal{A}}^{i} \in \widehat{\Upsilon}^{i}$$

- And obtain the VI

$$\boxed{N\gamma_{\mathcal{E}}^{l+1} + \mathbf{r} \in -\mathcal{N}_{\Upsilon}(\gamma_{\mathcal{E}}) \; ; \; \gamma_{\mathcal{E}}^{l+1} \in \Upsilon}$$

- Where

$$N \equiv \left[D_{\mathcal{E}}{}^{T} M D_{\mathcal{E}} - E_{\mathcal{E}}\right]$$

$$\mathbf{r} \equiv D_{\mathcal{E}}{}^{T}\left(\mathbf{v}^{l} + hM^{-1}\mathbf{f}(\mathbf{q}, \mathbf{v}, t)\right) - \mathbf{c}$$

- And

$$E_{\mathcal{E}} = diag\left[E_{i}\right]; \quad E_{i} = -\left(h^{2}K_{i} + hR_{i}\right) \prec 0$$

$$\mathbf{c}^{i} = \left(h^{2}K^{i} + hR^{i}\right)^{-1}\left(\gamma_{\mathcal{A}}^{i,l} + hR^{i}(\dot{\mathbf{y}}^{l} - \dot{\mathbf{y}}_{P}^{l})\right)$$

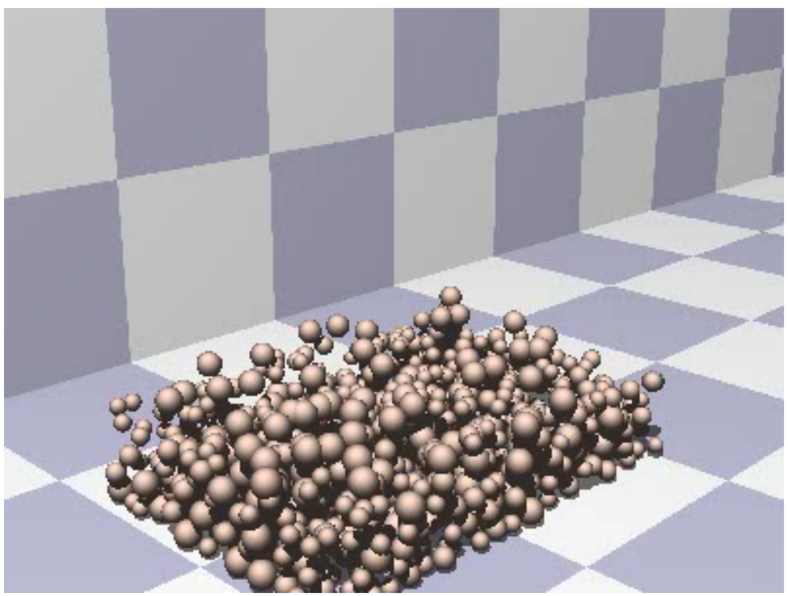
- Once we compute new contact impulse, we compute velocity, and update position, and iterate.

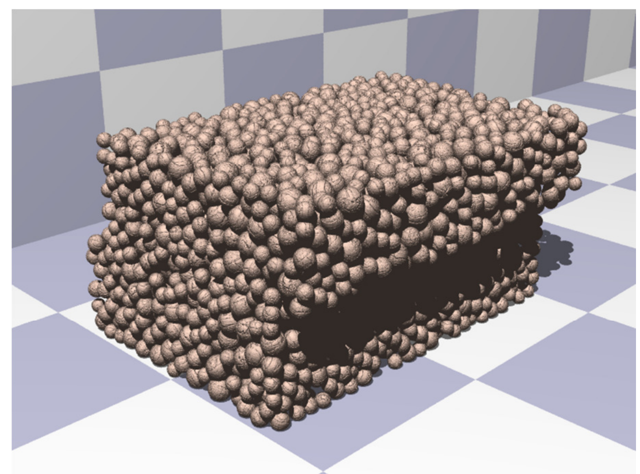
How do we solve it:

- It is the same scheme as before, just the projection on the new set needs to be computed and implemented.

Numerical Experiments

- Compaction and shear test of a granular media.
- Advanced cases of earth-moving machines (bulldozzers, vehicles) need such things to understand soil reaction.
- Configuration: soil sample put in 0.1m x 0.1 mx0.2 m,
- Top part is pulled by a “drawer” after a mass is dropped on top; shear force is measured and compared with experiment.



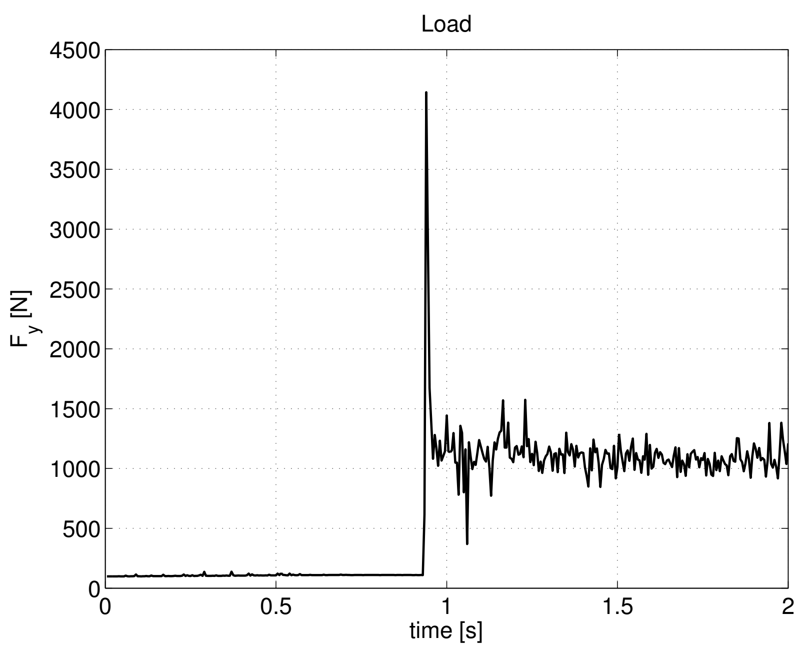


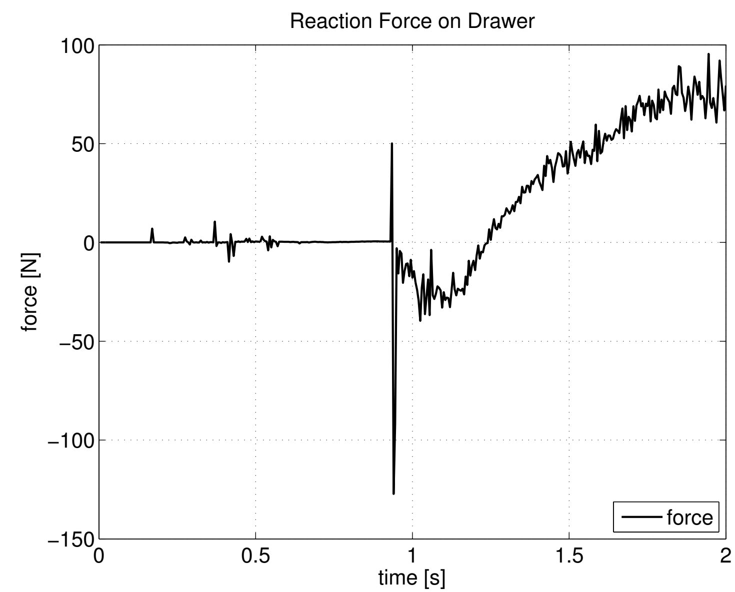
Results.

- Configuration: Getting parameters is hard; Normal and tangentia compliances are 0:910e-7 m/N and damping coefficient is = 0.1. Sphere distribution.

Number	Density	Diameter	Friction
90	1700 kg/m^3	0.020 m	0.5
630	1700 kg/m^3	0.010 m	0.5

- Evolution of the shear force and vertical force. These will be measured from experiments. Note spike in forces – including shear! When load is dropped (probably some lock-in before it relaxes and pushes on the drawe)





4. Rolling Friction Models

Components of the rolling friction

- The contact plane has a normal vector and consists of two tangential vectors

$$\mathbf{n}_i, \mathbf{u}_i, \mathbf{v}_i$$

- Force between bodies at contact, has a normal component and a tangential component

$$\mathbf{F}_{i,N} = \hat{\gamma}_{i,n}\mathbf{n}_i \quad \mathbf{F}_{i,T} = \hat{\gamma}_{i,u}\mathbf{u}_i + \hat{\gamma}_{i,w}\mathbf{w}_i$$

σ_i

- Torque between bodies at contact, has a normal component and a tangential component.

$$\mathbf{M}_{i,N} = \hat{\tau}_{i,n}\mathbf{n}_i \quad M_{i,T} = \hat{\tau}_{i,u}\mathbf{u}_i + \hat{\tau}_{i,w}\mathbf{w}_i$$

- Dual motion quantities: tangential velocity, normal rotation and tangential rotation: $\mathbf{v}_{i,T} \quad \omega_{i,N}, \quad \omega_{i,T}$

Model: rolling, spinning, sliding

- Sliding friction, friction coefficient μ_i

$$\mu_i \widehat{\gamma}_{i,n} \geq \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2} \ , \quad ||\boldsymbol{v}_{i,T}|| \left(\mu_i \widehat{\gamma}_{i,n} - \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2} \right) = 0,$$
$$\langle \boldsymbol{F}_{i,T}, \boldsymbol{v}_{i,T} \rangle = - ||\boldsymbol{F}_{i,T}|| \ ||\boldsymbol{v}_{i,T}||$$

- Rolling coefficient, ρ_i

$$\rho_i \widehat{\gamma}_{i,n} \geq \sqrt{\widehat{\tau}_{i,u}^2 + \widehat{\tau}_{i,w}^2} \ , \quad ||\boldsymbol{\omega}_{i,T}|| \left(\rho_i \widehat{\gamma}_{i,n} - \sqrt{\widehat{\tau}_{i,u}^2 + \widehat{\tau}_{i,w}^2} \right) = 0,$$
$$\langle \boldsymbol{M}_{i,T}, \boldsymbol{\omega}_{i,T} \rangle = - ||\boldsymbol{M}_{i,T}|| \ ||\boldsymbol{\omega}_{i,T}||$$

- Spinning: Spinning coefficient, σ_i

$$\sigma_i \widehat{\gamma}_{i,n} \geq \widehat{\tau}_{i,n} \ , \quad ||\boldsymbol{\omega}_{i,N}|| \left(\sigma_i \widehat{\gamma}_{i,n} - \widehat{\tau}_{i,n} \right) = 0,$$
$$\langle \boldsymbol{M}_{i,N}, \boldsymbol{\omega}_{i,N} \rangle = - ||\boldsymbol{M}_{i,N}|| \ ||\boldsymbol{\omega}_{i,N}||$$

Restating the model to expose the maximum dissipation principle

- This exposes the conic constraints:

$$
\begin{aligned}
(\widehat{\tau}_{i,u}, \widehat{\tau}_{i,w}) \;=\; & \operatorname{argmin}\; \boldsymbol{\omega}^T \left(\boldsymbol{D}_{\tau_u}\widehat{\tau}_{i,u} + \boldsymbol{D}_{\tau_w}\widehat{\tau}_{i,w}\right) \\
& \text{s.t.}\; \sqrt{\widehat{\tau}_{i,u}^2 + \widehat{\tau}_{i,w}^2} \le \rho\widehat{\gamma}_{i,n}
\end{aligned}
$$

$$
\begin{aligned}
(\widehat{\tau}_{i,n}) \;=\; & \operatorname{argmin}\; \boldsymbol{\omega}^T \left(\boldsymbol{D}_{\tau_n}\widehat{\tau}_{i,n}\right) \\
& \text{s.t.}\; \left|\widehat{\tau}_{i,n}\right| \le \sigma\widehat{\gamma}_{i,n}
\end{aligned}
$$

$$
\begin{aligned}
(\widehat{\gamma}_{i,u}, \widehat{\gamma}_{i,w}) \;=\; & \operatorname{argmin}\; \boldsymbol{v}^T \left(\boldsymbol{D}_{\gamma_u}\widehat{\gamma}_{i,u} + \boldsymbol{D}_{\gamma_w}\widehat{\gamma}_{i,w}\right) \\
& \text{s.t.}\; \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2} \le \mu\widehat{\gamma}_{i,n}
\end{aligned}
$$

Time-stepping with convex relaxation.

- The reaction cone:

$$\mathcal{Z}^i = \left\{ \gamma \in \mathbb{R}^6 \mid \mu^i \gamma_n \geq \sqrt{\gamma_u^2 + \gamma_w^2}, \rho^i \gamma_n \geq \sqrt{\tau_u^2 + \tau_w^2}, \sigma^i \gamma_n \geq |\tau_n| \right\}$$

- Define total cone (inclusive of bilateral constraints) and its polar:

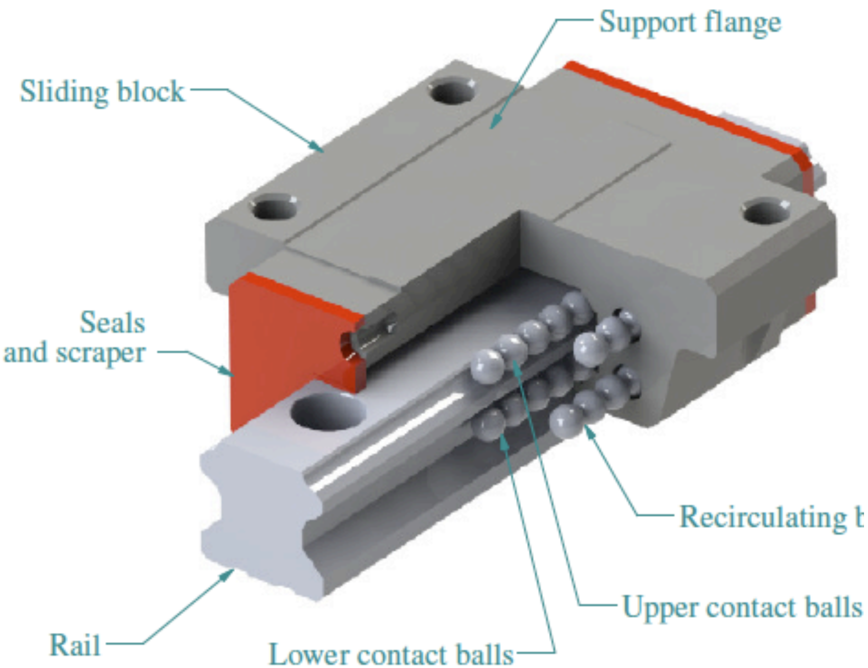
$$\Upsilon = \left(\times_{i \in \mathcal{G}_\mathcal{A}} \mathcal{Z}^i \right) \times \left(\times_{i \in \mathcal{G}_\mathcal{B}} \mathcal{B}^i \right)$$
$$\Upsilon^\circ = \left(\times_{i \in \mathcal{G}_\mathcal{A}} \mathcal{Z}^{i^\circ} \right) \times \left(\times_{i \in \mathcal{G}_\mathcal{B}} \mathcal{B}^{i^\circ} \right)$$

- Using the virtually identical sequence of time stepping, notations and relaxation we obtain the cone complementarity problem.

$$(N \boldsymbol{\gamma}_\mathcal{S} + \boldsymbol{r}) \in -\Upsilon^\circ \quad \perp \quad \boldsymbol{\gamma}_\mathcal{S} \in \Upsilon.$$

Validation: linear guideway with recirculating balls

- We get close matching to analytical result (we allow a bit of compliance to resolve indeterminacy).



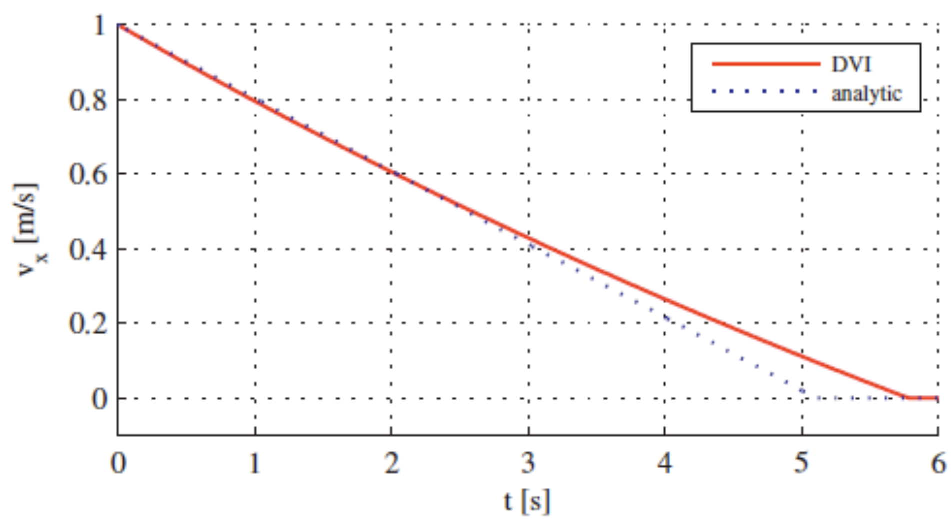
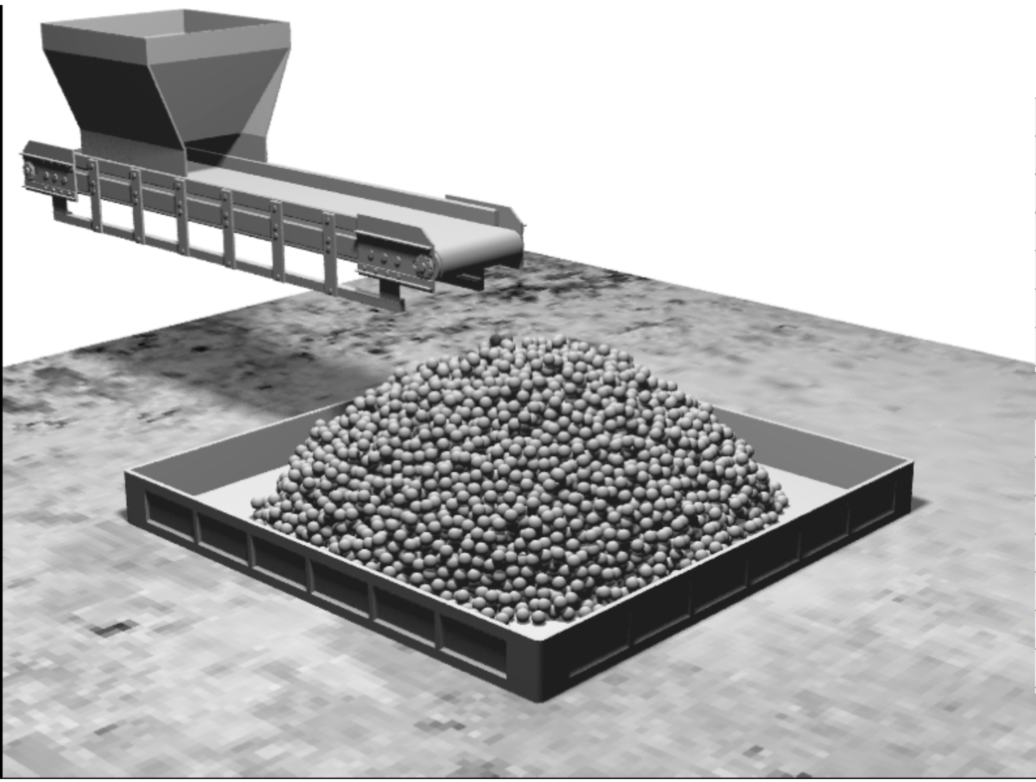
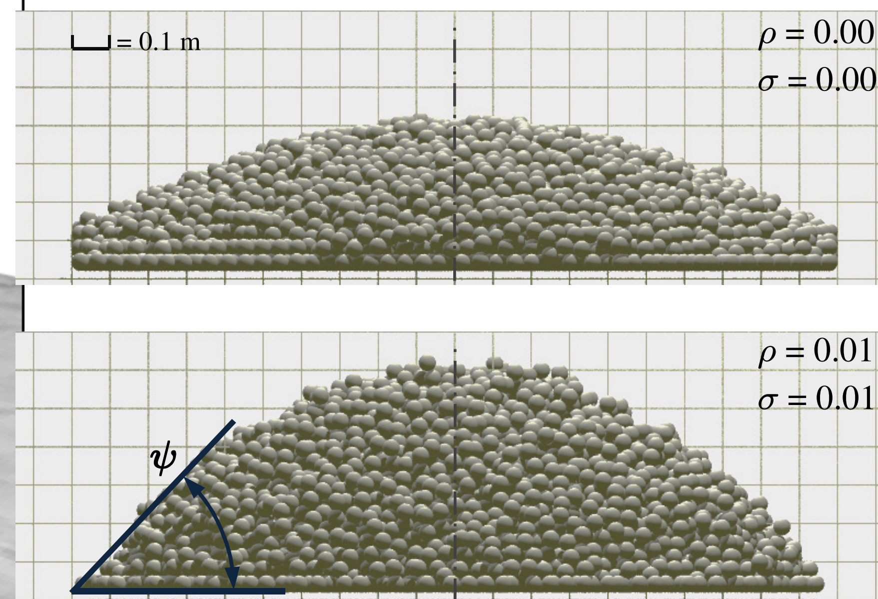


Fig. 3. Velocity of the rolling disk: comparison with classic theory.

Rolling Friction Effects on Granular Materials

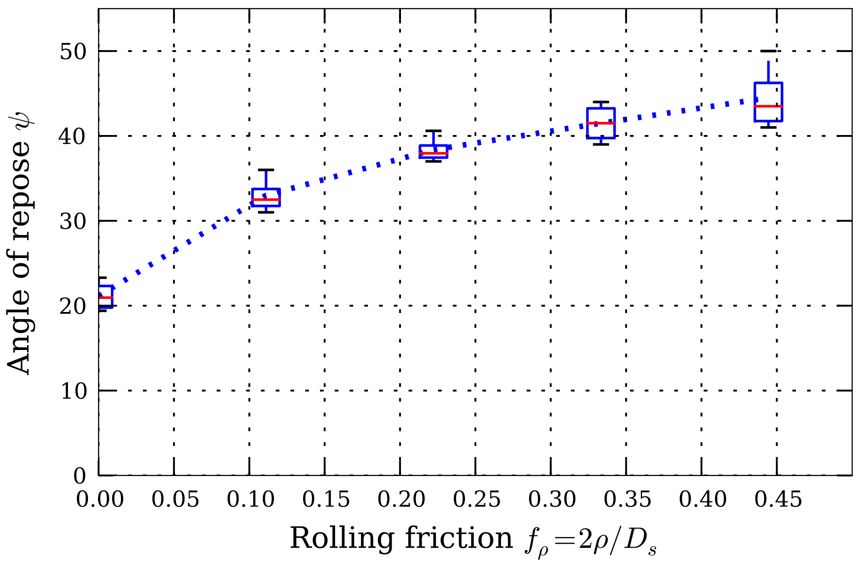
- Converyor belt scenario: Question can we reproduce rolling friction effects over the repose angle of piles (observed)?
- Yes, for same sliding friction the results are different.

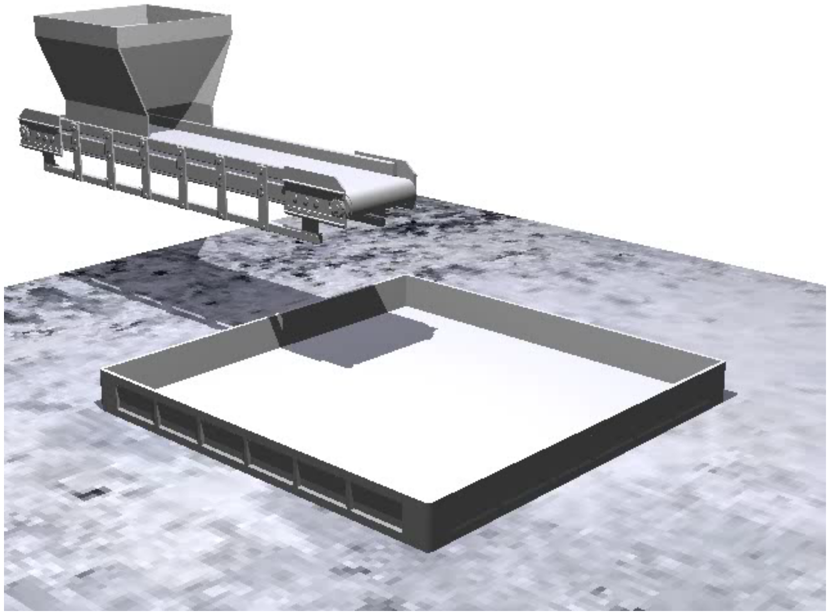




Angle of repose.

- Dependence on rolling friction coefficient – confirms trends in other experiments.





Conclusions

- Relaxation we use to obtain tractable subproblems.
- This allows a flexible modeling of contact behavior by changing the constraint set for the forces/impulses.

- We presented extensions of (DVI) time-stepping schemes to rolling contact and elasto-plastic contact .
- These include convex relaxation and algorithmic considerations.
- The method is implemented in ChronoEngine.