Complementary Variables

Are variables that satisfy

$$s \geq 0,\ x \geq 0,\ s^T x = 0 \leftrightarrow 0 \leq s \perp x \geq 0$$

Their most common occurence is perhaps in the optimality conditions of problems with bound constraints

$$min_{x \geq 0} F(x) \Rightarrow \nabla_x F(x) - s = 0,\ 0 \leq s \perp x \geq 0$$

But their modeling power exceeds optimization since **they can quantify alternatives**.

(Linear) Complementary Problems, (LCP)

$$s = \mathcal{M}x + q(F(x)),\ s \geq 0,\ x \geq 0,\ s^T x = 0.$$

- Examples: Linear and Quadratic Programming.
- Important classes of matrices: **PSD** $(x^T \mathcal{M} x \geq 0, \forall x)$ and **copositive** $(x^T \mathcal{M} x \geq 0, \forall x \geq 0)$.
- **LCP**'s involving copositive matrices do not have a solution in general.
- Let $\mathcal{M}$ be copositive. If, $x \geq 0$ and $x^T \mathcal{M} x = 0$ implies $q^T x \geq 0$, then the **LCP** has a solution that can be found by Lemke's algorithm.

(Parameterized) Variational Inequalities ((P)VI)

Problem: Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$, $F \in \mathcal{C}^2$, and $\mathcal{K} \subset \mathbb{R}^m$ be a convex set. Find $y \in \mathbb{R}^m$ such that

$$\langle F(x,y), v - y \rangle \geq 0, \quad \forall v \in \mathcal{K}.$$

x are the design variables, y are the state variables. **Solution set of the variational inequality:** $\mathcal{S}(x)$.

Complementarity Constraint Formulation of Variational Inequality

Any Parameterized Variational Inequality **(PVI)** can be represented as a problem with complementarity constraints. If $\mathcal{K} = \{ v \in \mathbb{R}^m | \, v \geq b \}$, for some vector $b \in \mathbb{R}^m$, the parameterized variational inequality can be represented as

$$
\begin{aligned}
F(x,y) &\geq 0, \\
y &\geq b, \\
(y-b)^T F(x,y) &= 0.
\end{aligned}
$$

Complementarity Constraint Formulation of PVI

Any Parameterized Variational Inequality **(PVI)** can be represented as a problem with complementarity constraints. If $\mathcal{K} = \{ v \in \mathbb{R}^m | \, v \geq b \}$, for some vector $b \in \mathbb{R}^m$, the parameterized variational inequality can be represented as

$$\begin{aligned} F(x,y) &\geq 0, \\ y &\geq b, \\ (y-b)^T F(x,y) &= 0. \end{aligned}$$

Optimal Design of PVI

Design parameters x are required to be in set $\mathcal{F}$.

Variational Formulation

$$\begin{array}{rl} \min_{x,u} & \tilde{f}(x,u) \\ \text{subject to} & x \in \mathcal{F} \\ & u \in \mathcal{S}(x) \end{array}$$

Complementarity Formulation

$$\begin{array}{rll} \min_{x,u} & \tilde{f}(x,u) & \\ \text{subject to} & h_i(x) & = 0, i = 1,2,\ldots,n_h \\ & g_j(x) & \leq 0, j = 1,2,\ldots,n_g \\ & F(x,y) & \geq 0, \\ & y-b & \geq 0, \\ (y-b)^T & F(x,y) & = 0. \end{array}$$

For the obstacle problem, we have that $\nabla_y F(x,y)$ is positiv[e] definite for any value of x.

Mathematical Problems with Complementary Constraints (MPCC)

$$
\begin{array}{lll}
\text{minimize}_{x,y} & f(x,y) & \\
\text{subject to} & g(x,y) & \geq 0 \\
& h(x,y) & = 0 \\
& F(x,y) & = s \\
& y & \geq 0 \\
& s & \geq 0 \\
\text{Compl. constr.} & s^T y & = 0
\end{array}
$$

The functions f, F and h are smooth.

Example problems

- Contact problems (Robotics, Virtual reality, Structural engineering).
- Game theoretical based models (Economics, Energy Markets).
- Finance (Black Sholes, American Option).
- Bilevel Optimization, where the lower-level problem has inequality constraints (Economics, Energy Markets, Process Engineering).
- ...

Contact Problems, Formulation and Design

Discretization of elastic membrane with rigid obstacle, defined by the mapping $\chi : \Omega(x) \to \mathbb{R}$, $\Omega(x) \subset \mathbb{R}^2$. x **are the design parameters.** Define

$$\begin{aligned} \mathcal{K} &= \left\{ v \in H_0^1(\Omega(x)) \,\middle|\, v \geq \chi \text{ a.e. in } \Omega(x) \right\} \\ F(x,u) &= -\Delta u - f \end{aligned}$$

where f is the force perpendicular to the membrane applied to each point (e.g. gravity).

Problem Find the shape of the membrane $u \in \mathcal{K}$ subject to the rigid obstacle constraint:

$$\langle F(x,u), v - u \rangle \geq 0, \quad \forall v \in \mathcal{K}.$$

Most free boundary problems can be expressed like PVI!

Nash Games

Nash Game: **non-cooperative equilibrium** of several producers

$$z_i^* \in \left\{ \begin{array}{ll} \underset{z_i}{\text{argmin}} & b_i(\hat{z}) \\ \text{subject to} & c_i(z_i) \geq 0 \end{array} \right. \qquad \text{producer } i$$

- $\hat{z} = (z_1^*, \ldots, z_{i-1}^*, z_i, z_{i+1}^*, \ldots, z_l^*)$
- All producers/players are equal

Complementarity Formulation

Introduce slacks s, and form optimality condtions ...

$$\begin{aligned} \nabla b(z) - \nabla c(z)\lambda &= 0 \\ s - c(z) &= 0 \\ 0 \leq \lambda \perp s &\geq 0 \end{aligned}$$

where

- $b(z) = (b_1(z), \ldots, b_k(z))$ & $c(z) = (c_1(z), \ldots, c_k(z))$
- $\perp$ means $\lambda^T s = 0$, either $\lambda_i > 0$ or $s_i > 0$

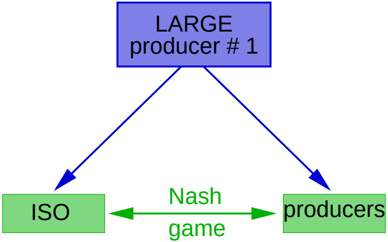
$$\left.\begin{array}{l} y = (z, \lambda, s) \\ h = \ldots \end{array}\right\} \quad \ldots \text{ becomes } \ldots \quad \left\{\begin{array}{l} h(y) = 0 \\ 0 \leq y_1 \perp y_2 \geq 0 \end{array}\right.$$

- **Nonlinear complementarity problem** (NCP)
- **Robust large scale solvers** exist: `PATH`

Stackelberg Games

Single dominant producer & Nash followers

$$\begin{cases} \min_{x \geq 0, y} & f(x,y) \\ & h(x,y) = 0 \\ & 0 \leq y_1 \perp y_2 \geq 0 \end{cases}$$



- Nash game ($h(x, y) = 0$) parameterized in leader's variables x
- **Mathematical Program with Complementarity Constraints (MPCC)**
- For the Penn-NJ-Mar NOx allowance market **MPCC with 20,000 vars and 10,000 cons**!

Complexity Issues

- LCP with PSD matrices can be solved in polynomial time.
- General LCP problems have currently known worst-case exponential complexity.
- Nonlinear CP are harder than linear CP (iterative LCP).
- MPCC are harder than Nonlinear CP (iterative LCP).
- Differential Problems with Complementarity Constraints may be even harder, but their discretization leads to one of the problems above. Nonetheless, they pose serious convergence difficulties (as the time step or mesh size go to zero), but we will not address those today.

"Hottest" algorithms

- Lemke's algorithm: simplex-like pivotal algorithm, excellent robustness, complexity may increase quickly. Implemented in PATH.
- Interior point algorithms, where the regularization parameter ϵ is decreased driven down to 0 after a few inner iterations

$$s = \mathcal{M}x + q,\ s \geq 0,\ x \geq 0,\ s^T x = 0 \Rightarrow s = \mathcal{M}x + q, sx = \epsilon$$

Works for PSD matrices only

- Semismooth approaches : rephrase the complementarity constraints with a semismooth function (such as Fischer-Burmeister) and use semismooth Newton method.

Nonsmooth approach for MPCC

- Applies to the variational approach. If the variational inequality is regular, then $\mathcal{S}(x)$ contains only one point and defines a continuous mapping $y(x)$.
- However, $y(x)$ is **nondifferentiable**, due to the change of the active set with x.
- Need to use generalized gradients in a bundle trust-region method to solve

$$\begin{array}{rc} \min & f(x, y(x)) \\ \text{subject to} & x \in \mathcal{F} \end{array}$$

- **Problem: May need a number of computations that grows exponentially in the number of degenerate pairs**
- **Software for large-scale problems is inexistent**

Nonlinear Programming Approach for MPCC

Solve the complementarity formulation by a nonlinear programming approach. **Problem: The feasible set has no relative interior (later), therefore neither will its linearization, because of the complementarity constraints: No constraint qualification**.

$$x \leq 0,\ y \leq 0,\ xy = 0 \Rightarrow x, y \text{ cannot both be negative}$$

Major problem for most algorithms based whether based on linearization (Sequential Quadratic Programming) or not. **Need algorithms that accomodate this type of degeneracy, since all classical algorithms assume that a constraint qualification holds**.

Nonlinear Program (NLP)

For $f, g, h \in \mathcal{C}^2(R^n)$

$$\begin{array}{lll} \text{minimize}_{x\in\mathbb{R}^n} & f(x) & \\ \text{subject to} & h_i(x) = 0 & i = 1, ..., r \\ & g_j(x) \leq 0 & j = 1, ..., m \end{array}$$

Inequality Constraints Only

$$\begin{array}{lll} \text{minimize}_{x\in\mathbb{R}^n} & f(x) & \\ \text{subject to} & g_j(x) \leq 0 & j = 1, ..., m \end{array}$$

Any results can be extended for equality constraints as long as $\nabla_x h_i(x)$, $i = 1, \ldots, r$ are linearly independent.

"Steepest Descent" for Nonlinearly Constrained Optimization

SQP: **Sequential Quadratic Programming**.

1. Set $k = 0$, choose x^0.
2. Compute d^k from

$$\begin{array}{ll} \text{minimize} & \nabla f(x^k)^T d + \frac{1}{2} d^T d \\ & g_j(x^k) + \nabla g_j(x^k)^T d \leq 0, \qquad j = 1, \ldots, m. \end{array}$$

3. Choose α^k using Armijo for the nondifferentiable merit function $\phi(x) = f(x) + c_\phi \max\{g_0(x), g_1(x), \ldots g_m(x), 0\}$, $c_\phi > 0$, and set $x^{(k+1)} = x^k + \alpha^k d^k$.
4. Set $k = k + 1$ and return to Step 2.

Optimality Conditions

Assume $f(x)$ is twice differentiable. Problem:

$$\text{minimize}_{x \in \mathbb{R}^n} \qquad f(x)$$

First-order necessary conditions: (nonlinear equations)

$$\nabla_x f(x) = 0$$

Second-order necessary conditions

$$\nabla^2_{xx} f(x) \succeq 0$$

Second-order sufficient conditions (Newton iteration): FOC

$$\nabla^2_{xx} f(x) \succ 0$$

Mangasarian-Fromovitz Constraint Qualification

- **Mangasarian Fromovitz CQ (MFCQ)**: The tangent cone to the feasible set $\mathcal{T}(x^*)$ has a nonempty interior at a solution x^* or

$$\exists p \in R^n \;;\; \text{such that } \nabla_x g_j(x^*)^T p < 0,\; j \in \mathcal{A}(x^*).$$

- MFCQ holds $\Leftrightarrow$ The set $\mathcal{M}(x^*)$ of the multipliers satisfying KKT is nonempty and bounded.
- The **critical cone**:

$$\mathcal{C} = \{u \in \mathbb{R}^n \,|\, \nabla_x g_j(x^*)^T u \le 0,\; j \in \mathcal{A}(x^*),\; \nabla_x f(x^*)^T u \le 0\}$$

- If MFCQ does not hold then $\mathcal{T}(x,u) = \{u \in \mathbb{R}^n, \,|\, g_j(x) + \nabla_x g_j(x)^T u \le 0,\; j = 1, \ldots, m\}$ may be empty x arbitrarily close to x^*. **Problem for SQP!**

Karush Kuhn Tucker (KKT) conditions

The active set at a feasible $x \in \mathbb{R}^n$:

$$\mathcal{A}(x) = \{j | 1 \leq j \leq m,\ g_j(x) = 0\}$$

Stationary point of NLP : A point x for which there exists $\lambda \geq 0$ such that

$$\nabla_x f(x) + \sum_{j \in \mathcal{A}(x)} \lambda_j \nabla_x g_j(x) = 0$$

The Lagrangian:
$\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) = f(x) + \lambda^T g(x)$.
Complementarity formulation for stationary point:
$\emptyset \neq \mathcal{M}(x) =$
$\left\{\lambda \in \mathbb{R}^m \,|\, \lambda \geq 0,\ \nabla_x \mathcal{L}(x, \lambda) = 0,\ g(z) \leq 0,\ (\lambda)^T g(z) = 0\right\}$
KKT theorem: MFCQ $\Rightarrow$ the solution x^* of the NLP is a

Second-order optimality conditions (SOC)

Sufficient SOC: MFCQ and $\exists \tilde{\sigma} > 0$ such that $\forall u \in \mathcal{C}(x^*)$

$$\max_{\lambda \in \mathcal{M}(x^*)} u^T \mathcal{L}_{xx}(x^*, \lambda) u = \max_{\lambda \in \mathcal{M}(x^*)} u^T \nabla^2_{xx}(f + \lambda^T g)(x^*) u \geq \tilde{\sigma} \|u\|^2.$$

$$\left(\exists \lambda \in \mathcal{M}(x^*) \quad u^T \mathcal{L}_{xx}(x^*, \lambda) u = u^T \nabla^2_{xx}(f + \lambda^T g)(x^*) u > \tilde{\sigma} \|u\|^2 \right)$$

Sufficient SOC imply **Quadratic Growth:**

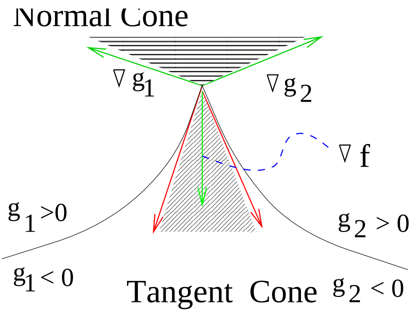
$$\max \{ f(x) - f(x^*), g_1(x), g_2(x), \ldots, g_m(x) \} \geq \sigma \|x - x^*\|^2 > 0$$

MFCQ + Quadratic Growth $\Rightarrow$ x^* is an isolated stationary point and certain SQP algorithms will achieve at least loca linear convergence

Lack of Constraint Qualification

Problem: The feasible set has no relative interior (later), therefore neither will its linearization, because of the complementarity constraints: No constraint qualification.

$x \leq 0,\ y \leq 0,\ xy = 0 \Rightarrow x, y$ **cannot both be negative**



Formulation

$$\begin{aligned} \min_x \quad & f(x) \ \text{subject to} \\ & g(x) \geq 0, \quad h(x) = 0, \\ & 0 \leq G^T x \perp H^T x \geq 0, \end{aligned}$$

where

- G and H are $n \times m$ column submatrices of the $n \times n$ identity matrix (with no columns in common): *lower bounds*;
- $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, and $h : \mathbb{R}^n \to \mathbb{R}^q$ are twice continuously differentiable. **Think** $G^T x = x_1 (= y)$**,** $H^T x = x_2 (= s)$

Theory extends to nonlinear functions $0 \leq G(x) \perp H(x) \geq 0$. We use bounds because they can be enforced explicitly by algorithms for the NLP subproblem; this leads to some nice properties.

Some Definitions

Stationarity for MPCC at a feasible point x^*: Define active sets:

$$
\begin{aligned}
I_g &\triangleq \{i \in \{1,2,\ldots,p\} \mid g_i(x^*) = 0\},\\
I_G &\triangleq \{i \in \{1,2,\ldots,m\} \mid G_i^T x^* = 0\},\\
I_H &\triangleq \{i \in \{1,2,\ldots,m\} \mid H_i^T x^* = 0\},
\end{aligned}
$$

Feasibility $\Rightarrow$
$I_G \cup I_H = \{1,2,\ldots,m\}$

Multiplier tuple (λ,μ,τ,ν) defines **MPCC Lagrangian** :

$$L(x,\lambda,\mu,\tau,\nu) = f(x) - \lambda^T g(x) - \mu^T h(x) - \tau^T G^T x - \nu^T H^T x.$$

Constraint qualifications: **MPCC-LICQ:** $\mathcal{K}$ is linearly independent set (ensures that $(\lambda^*,\mu^*,\tau^*,\nu^*)$ satisfying stationarity is unique):

$$\mathcal{K} \triangleq \{\nabla g_i(x^*)\}_{i\in I_g} \cup \{\nabla h_i(x^*)\}_{i=1,2,\ldots,q} \cup \{G_i\}_{i\in I_G} \cup \{H_i\}_{i\in I_H}.$$

Second-Order Conditions for MPCC

MPCC-SOSC: Let x^* be strongly stationary. There is $\sigma > 0$ such that for every $s \in S^*$, there are multipliers such that

$$s^T \nabla^2_{xx} L(x^*, \lambda^*, \mu^*, \tau^*, \nu^*) s \geq \sigma \|s\|^2.$$

Stationary points satisfy ...

$$
\begin{aligned}
\nabla_x L(x^*, \lambda^*, \mu^*, \tau^*, \nu^*) &= 0, \\
0 \leq \lambda^* \perp g(x^*) &\geq 0, \\
h(x^*) &= 0, \\
\tau^* \perp G^T x^* &\geq 0, \\
\nu^* \perp H^T x^* &\geq 0,
\end{aligned}
$$

...**AND**, from stronger to weaker concept, .

- Strong stationarity: $\tau_i^* \geq 0 \;\; \nu_i^* \geq 0, \;\; i \in I_G \cap I_H,$.
- M-stationarity: $\tau_i^* \nu_i^* \geq 0$ but not both τ_i^*, ν_i^* negative, for $i \in I_G \cap I_H$.
- C-stationarity: $\tau_i^* \nu_i^* \geq 0$ for $i \in I_G \cap I_H$.

Strong stationarity: there is no direction that decreases f

Typical assumptions for MPCC analysis

- MPCC-LICQ
- MPCC-SOSC
- Both are **generic** at a solution.

Idea explained on NLP reformulation: Assumptions

$\min_x f(x)$ subject to $g_i(x) \leq 0, \quad i = 1, 2, \ldots m.$

- The Lagrange Multiplier set is not empty (but may be unbounded).
- The quadratic growth condition holds

$$\max \{f(x) - f(x^*), g_1(x), g_2(x), \ldots, g_m(x)\} \geq \sigma \|x - x^*\|^2$$

- f, g are twice continuously differentiable.
- Note that quadratic growth is the weakest possible second-order condition!

Idea explained on NLP reformulation: Elastic Mode Reformulation

$\min_{x,\zeta} \; f(x) + c\zeta \;$ subject to $\; g_i(x) \leq \zeta, \;\; i = 1, 2, \ldots m, \;\; \zeta \geq 0.$
For $c > c_\zeta$ at $(x^*, 0)$ we have

- The Lagrange multiplier set is nonempty and bounded (**MFCQ holds**).
- The quadratic growth condition is satisfied.
- The data of the problem are twice differentiable.
- Linear Convergence of NLP formulation of MPCC is guaranteed for Steepest Descent (**MA05**).
- Quadratic Convergence is achievable under certain assumptions (**MA05**).

Effect of the Modification

- **Lagrange Multiplier set of the original problem**: $\mathcal{M}(x^*)$.
- **Lagrange Multiplier set of the modified problem**: $\mathcal{M}^c(x^*, 0)$. $\mu^c \in \mathcal{M}^c(x^*, 0) \Rightarrow \|\mu^c\| = c$.
- **Reduced Lagrange Multiplier set**.

$$\begin{aligned}\mathcal{M}_r^c(x^*) \quad = \quad & \{\mu \in \mathbb{R}^m \mid \exists \mu_{m+1} \in \mathbb{R}, \\ & \text{such that } \ (\mu, \mu_{m+1}) \in \mathcal{M}^c((x^*, 0))\}.\end{aligned}$$

- $\mathcal{M}_r^c(x^*) \subset \mathcal{M}(x^*)$. The penalty term $c\zeta$ has the effect of preserving only the multipliers $\mu \in \mathcal{M}(x^*)$ with $\|\mu\|_1 \leq c$!

Numerical Experiments with SNOPT

The elastic mode of SNOPT implements a similar approach. Runs done on NEOS for the MacMPCC collection.

Problem	Var-Con-CC	Value	Status	Feval	Elastic
gnash14	21-13-1	-0.17904	Optimal	27	Yes
gnash15	21-13-1	-354.699	Optimal	12	None
gnash16	21-13-1	-241.441	Optimal	7	None
gnash17	21-13-1	-90.7491	Optimal	9	None
gne	16-17-10	0	Optimal	10	Yes
pack-rig1-8	89-76-1	0.721818	Optimal	15	None
pack-rig1-16	401-326-1	0.742102	Optimal	21	None
pack-rig1-32	1697-1354-1	0.751564	Optimal	19	None

Results Obtained with MINOS

Runs done with NEOS for the MacMPCC collection.

Problem	Var-Con-CC	Value	Status	Feval	Infeas
gnash14	21-13-1	-0.17904	Optimal	80	0.0
gnash15	21-13-1	-354.699	Infeasible	236	7.1E0
gnash16	21-13-1	-241.441	Infeasible	272	1.0E1
gnash17	21-13-1	-90.7491	Infeasible	439	5.3E0
gne	16-17-10	0	Infeasible	259	2.6E1
pack-rig1-8	89-76-1	0.721818	Optimal	220	0.0E0
pack-rig1-16	401-326-1	0.742102	Optimal	1460	0.0E0
pack-rig1-32	1697-1354-1	N/A	Interrupted	N/A	N/A

What is NLP global convergence?

It is global convergence to a **local** stationary point. That is, the algorithm does not get lost at points that have no relevance to the original problem. It means that the algorithm either.

1. Terminates at an infeasible point (which is a minimizer of some norm of the constraint infeasibility). .
2. It terminates at a feasible point that does not satisfy a constraint qualification.
3. Terminates at a stationary point.

Elastic Formulation: Global Convergence

$$
\text{Elastic}(c): \quad \min_{x,\zeta} \; f(x) + c\zeta + c(G^T x)^T (H^T x) \;\text{ subject to}
$$
$$
g(x) \geq -\zeta e_p, \;\; \zeta e_q \geq h(x) \geq -\zeta e_q, \;\; 0 \leq \zeta \leq \bar{\zeta},
$$
$$
G^T x \geq 0, \;\; H^T x \geq 0,
$$

Sequence of Inexact First-Order Points

Given a sequence of inexact first-order points for Elastic(c_k), any accumulation point satisfying feasibility and CQ for the MPCC is C-stationary. Formally:

Theorem

$\{c_k\}$ positive, nondecreasing; $\{\epsilon_k\}$ is nonnegative with $\{c_k \epsilon_k\} \to 0$; (x^k, ζ_k) is an ϵ_k-first-order point of Elastic (c_k). If x^* is an accumulation point of $\{x^k\}$ that is feasible for MPCC and satisfies MPCC-LICQ, then x^* is C-stationary and $\zeta_k \to 0$ for the convergent subsequence.

Sequence of Exact Second-Order Points – answer to Q4

$\{c_k\} \uparrow \infty$ and let each (x^k, ζ_k) be a second-order point for Elastic(c_k).

Theorem

Either there is finite termination at some c_k (with x_k feasible for MPCC), or else any accumulation point of $\{x^k\}$ is infeasible for MPCC or else fails to satisfy MPCC-LICQ.

Proof: First show $\zeta_k = 0$ for k sufficiently large. Then if $(G_j^T x^*)(H_j^T x^*) > 0$ for some j and accumulation point x^*, can identify a direction of arbitrarily negative curvature over the subsequence of k's. (Contradicts second-order assumption.) **Key: Finite exact complementarity $\Rightarrow$ c_k is fixed for $k \geq k_0$**

Other convergence properties are corollaries of the *inexact* case.

Sequence of Inexact Second-Order Points – answer Q4

(x^k, ζ_k) is an (ϵ_k, δ_k)-second-order point of Elastic(c_k).

Theorem

Let $\{c_k\}$ nondecreasing, $\{\epsilon_k\}$ has $\{c_k \epsilon_k\} \to 0$, and $\{\delta_k\} \to 0$. Assume that acc point x^* is feasible for MPCC, satisfies MPCC-LICQ. Then have c^* such that if $c_k \geq c^*$, k large, we have

(a) *x^* is M-stationary for MPCC.*

(b) *$\{c_k\}$ bounded $\Rightarrow$ x^* strongly stationary for MPCC.*

(c) *$\tau^k \perp G^T x^k$ and $\nu^k \perp H^T x^k$ $\Rightarrow$* **finite exact complementarity** *$(G^T x^k)^T (H^T x^k) = 0$ (for k with x_k near x^* and $c_k \geq c^*$).*

Finite Exact Complementarity: Another Condition

Definition

The strengthened MPCC-LICQ (MPCC-SLICQ) holds at a feasible point x^* of MPCC if the vectors in each of the following sets are linearly independent:

$$\mathcal{K} \cup \{H_j\}, \ \text{for } j \in I_G \backslash I_H, \quad \mathcal{K} \cup \{G_j\}, \ \text{for } j \in I_H \backslash I_G,$$

where $\mathcal{K}$ is the usual set of active constraint gradients for MPCC.

Under similar conditions to the previous theorem, with MPCC-SLICQ replacing $\tau^k \perp G^T x^k$ and $\nu^k \perp H^T x^k$, get finite exact complementarity.

Algorithm Elastic-Inexact

Choose $c_0 > 0$, $\epsilon_0 > 0$, $M_\epsilon > M_c > 1$, and positive sequences $\{\delta_k\} \to 0$, $\{\omega_k\} \to 0$;

for $k = 0, 1, 2, \ldots$

find an (ϵ_k, δ_k)-second-order point (x^k, ζ_k) of PF(c_k) with Lagrange multipliers $(\lambda^k, \mu^{-k}, \mu^{+k}, \tau^k, \nu^k, \pi^{-k}, \pi^{+k})$;

if $\zeta_k + (G^T x^k)^T (H^T x^k) \geq \omega_k$,

set $c_{k+1} = M_c c_k$;

else

set $c_{k+1} = c_k$;

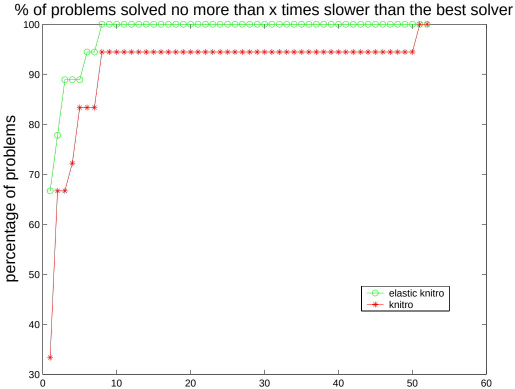
end (if)

choose $\epsilon_{k+1} \in (0, \epsilon_k / M_\epsilon]$.

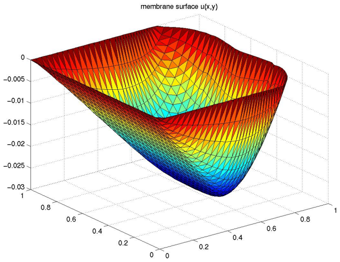
end (for)

Implementation issues

The elastic mode can be wrapped around any solver with good results (KNITRO)!



Problem


membrane surface u(x,y)
0
−0.005
−0.01
−0.015
−0.02
−0.025
−0.03
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

Run parameters

- Incident set identification (`is`)
- Packaging problem with pliant obstacle (`pc`)
- Packaging problem with rigid obstacle (`pr`).

Implement Elastic-Inexact using filterSQP (Fletcher/Leyffer) as the NLP solver, 6 examples per configuration (3 meshes times 2 types of obstacles). Parameters $c_0 = 10$, $\epsilon_0 = 10^{-3}$, $M_\epsilon = 15$, $M_c = 10$, $\omega_k = \min\{(k+1)^{-1}, c_k^{-1/2}\}$.

Aim: *Observe various features of the analysis: finite exact complementarity, second-order points for Elastic(c_k) at at the limiting MPCC, constraint qualifications.*

Means: Used AMPL scripts for implementation, dumped the derivative information on disk using "option auxfiles rc" and loaded it in Matlab using routines developped by Todd Munson.

Thanks to Sven Leyffer, Todd Munson.

Exact complementarity is satisfied at final point for all problems.

3/16 abnormal terminations:

Problem	Termination Message	c_k	Infeas
is-1-8	Optimal	10	
is-1-16	Optimal	10	
is-1-32	Small Trust Region	10	2.25e–07
is-2-8	Optimal	10	
is-2-16	Optimal	10	
is-2-32	Optimal	10^3	
pc-1-8	Optimal	10	
pc-1-16	Optimal	10^2	
pc-1-32	Optimal	10^3	
pc-2-8	Optimal	10^2	
pc-2-16	Optimal	10^5	
pc-2-32	Local Inf	10^4	6.06e–12
pr-1-8	Optimal	10^2	
pr-1-16	Optimal	10^3	
pr-1-32	Optimal	10^6	
pr-2-8	Optimal	10^2	
pr-2-16	Optimal	10^5	
pr-2-32	Local Inf	10^6	5.68e–13

Validates our **early satisfaction of exact complementarity** .

Constraint Qualifications and Second-Order Conditions

Define "numerically active" constraints using a tolerance of $\delta = 10^{-6}$.

Define J_{act} to be the matrix of numerically active constraint gradients.

For Q_2 spanning the null space of J_{act}, we measure satisfaction of MPCC-SLICQ via

$$\chi_{\mathrm{span}} \triangleq \min\left(\min_{i \notin I_G} \| Q_2^T G_i \|_2, \min_{i \notin I_H} \| Q_2^T H_i \|_2 \right),$$

Satisfaction of second-order conditions measured by examining eigenvalues of $Q_2^T L Q_2$, where L is Hessian Lagrangian at the last NLP.

Problem	n_F	m_{act}	$\text{cond}_2(J_{\text{act}})$	χspan	$\lambda_{\min}\left(Q_2^T H Q_2\right)$
is-1-8	193	181	3.45e+03	1.95e–03	0
is-1-16	763	742	4.39e+04	6.84e–04	0
is-1-32	3042	3020	5.26e+05	3.90e–09	0
is-2-8	184	180	2.17e+03	5.66e–04	1.08e–04
is-2-16	750	745	6.46e+04	8.44e–05	4.10e–07
is-2-32	3032	3025	∞	0	-1.48
pc-1-8	228	228	1.96e+02	0	∞
pc-1-16	970	964	9.38e+03	1.91e–06	5.55e–02
pc-1-32	3997	3972	4.48e+04	1.22e–08	4.88e–01
pc-2-8	233	228	3.40e+03	1.27e–04	1.37e+00
pc-2-16	977	964	1.34e+04	4.34e–06	6.62e–01
pc-2-32	4001	3972	7.82e+04	7.61e–09	2.06e–01
pr-1-8	186	179	1.10e+03	2.96e–17	2.61e–07
pr-1-16	754	739	4.11e+03	1.35e–18	0
pr-1-32	3040	3011	8.99e+07	3.56e–19	4.34e–01
pr-2-8	185	179	3.22e+03	1.47e–18	4.88e–01
pr-2-16	743	739	3.07e+03	1.91e–23	2.12e–01
pr-2-32	3027	3011	7.62e+03	8.92e–24	1.79e–01

MPCC-LICQ is 15/16, approx second-order point 16/16, MPCC-SLICQ 10/16.

Conclusions

- The elastic mode algorithms is a robust way of easily adapting your favorite solver to solve MPCC.
- We have theory for it in the case of MPCC, concerning local, global and rate of convergence results (MA05, MA06, MA, Tseng and Wright, in press).
- It has been adopted in several industrial strength codes IPOPT-C, LOQO, KNITRO in one variant or another (beyond SNOPT, that already had it).

Outstanding issues

- Can we show global convergence for inexact solves with fewer assumptions?
- Can we extend the global convergence results for interior point algorithms, or can we say something about the quality of the second-order points produced by interior point algorithms?
- Can we create a preconditioned conjugate gradient variant for very large scale optimization?
- Can we efficiently account for uncertainty?

Emerging application areas

MPCC relevance is due to the fact that there is a lower level optimization problem due to a game theoretical model.

- Energy distribution.
- Homeland Security and Nonproliferation applications with same.
- Massive adaptive data classification.
- Network design and bandwitdth allocation in high performance data and transportation networks.
- Taxation policy.
- Toll pricing in transportation networks.

Stochastic Mathematical Programs with Complementarity Constraints

SMPCC is a problem of the following form

$$\begin{array}{lrl} \min_{x,(y)(\omega)_{\omega\in\Omega}} & E_\omega\left[f(x,y(\omega),\omega)\right] & \\ \text{subject to} & c^1(x,y(\omega),\omega) & \leq 0; \omega \in \Omega \\ & c^2(x,y(\omega),\omega) & = 0; \omega \in \Omega \\ & 0 \leq y_1(\omega) \perp F(x,y(\omega),\omega) & \geq 0; \omega \in \Omega, \end{array} \quad (1)$$

where Ω are the events of the probability space $(\Omega, \mathcal{F}, P)$; f, c^1, c^2, and F are smooth vector-valued functions; and y_1 is a subvector of components of y.