Global convergence of elastic mode approaches for a class of MPCC

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Context and goals

- Recently, there have been several approaches to solve Mathematical Programs with Complementarity Constraints **(MPCC)** by using nonlinear programming techniques **(General: Anitescu 2000 , Fletcher and al 2002;**, **Structured smoothing: Fukushima and Pang 1998, Scholtes 2002)**.
- However all of them are of the local type: If point is **sufficiently close** to a strongly stationary point that satisfies **some condition** then algorithm converges to that point.
- Global convergence: If algorithm is applied to a **problem class** then any accumulation point is a **? stationary point**. If the point satisfies **some condition** then it is a **?++ stationary point**.
- However, we need to restrict the problem class to get some significant results.

Before anything else: The mixed P property

Let $A \in R^{(n_c+l)\times n_c}$, $B \in R^{(n_c+l)\times n_c}$, and $C \in R^{(n_c+l)\times l}$. $[A\ B\ C]$ is mixed P partition if

$$\left.\begin{array}{r} 0 \neq (y, w, z) \in R^{2n_c+l}, \\ Ay + Bw + Cz = 0 \end{array}\right\} \Rightarrow \exists i,\ 1 \leq i \leq n_c, \text{ such that } y_i w_i > 0.$$

What is actually **needed** in this work (and is implied if $[A\ B\ C]$ is a mixed P partition), is

$$A^T\theta \leq 0,\ B^T\theta \leq 0,\ C^T\theta = 0 \quad \Rightarrow \theta = 0$$

Optimization of mixed P variational inequalities

(OMPV)

$$
\begin{array}{lll}
\min\limits_{x,y,w,z} & f(x,y,w,z) & \\
\text{sbj.to} & g(x) & \leq 0 \\
 & h(x) & = 0 \\
 & F(x,y,w,z) & = 0 \\
 & y, w & \leq 0 \\
 & y^T w & \leq 0
\end{array}
$$

(OMPV(c))

$$
\begin{array}{llll}
\min\limits_{x,y,w,z,\zeta_1,\zeta_2} & f(x,y,w,z)+ & c(\zeta_1+\zeta_2) \\
\text{sbj.to} & g(x) & \leq 0 \\
 & h(x) & = 0 \\
-\zeta_1 e_{n_c+l} \leq & F(x,y,w,z) & \leq \zeta_1 e_{n_c+l} \\
 & y, w & \leq 0 \\
 & y^T w & \leq \zeta_2 \\
 & \zeta_1, \zeta_2 & \geq 0
\end{array}
$$

We name the problem **OMPV** because of the **mixed P VI**:

$$F(x,y,w,z) = 0 \; y, w \leq 0 \; y^T w \leq 0$$

MPEC stationarity concepts

$$
\begin{array}{rcl}
\nabla_x f(x,y,w,z)^T + \nabla_x h(x)^T\lambda + & & \\
\nabla_x g(x)^T\mu + \nabla_x F(x,y,w,z)^T\theta & = & 0 \\
\nabla_y f(x,y,w,z)^T + \widehat{\eta}_y + \nabla_y F(x,y,w,z)^T\theta & = & 0 \\
\nabla_w f(x,y,w,z)^T + \widehat{\eta}_w + \nabla_w F(x,y,w,z)^T\theta & = & 0 \\
\nabla_z f(x,y,w,z)^T + \nabla_z F(x,y,w,z)^T\theta & = & 0 \\
g(x) \le 0, \mu \ge 0, h(x) = 0, g(x)^T\mu = 0 & & \\
F(x,y,z,w) = 0, y \le 0,\ w \le 0,\ y^T w = 0, & & \\
\sum_{k=1}^{n_c} y_k |\widehat{\eta}_{y,k}| = 0,\ \sum_{k=1}^{n_c} w_k |\widehat{\eta}_{w,k}| = 0 & &
\end{array}.
$$

MPEC stationarity concepts

- Weakly stationary points : no additional requirements.
- C-stationary points: $\widehat{\eta}_{y,k}\widehat{\eta}_{w,k} \geq 0,\ k = 1, 2, \ldots, n_c$:
- M-stationary points: C-stationary points and $\widehat{\eta}_{y,k} \geq 0$ or $\widehat{\eta}_{w,k} \geq 0,\ k = 1, 2, \ldots, n_c$
- B-stationary points, for which $d = 0$ is a solution of the linearized (OMPV) **except** $y^T w \leq 0$
- Strongly stationary points,

$$y_k = 0,\ w_k = 0 \Rightarrow \widehat{\eta}_{y,k} \geq 0 \text{ and } \widehat{\eta}_{w,k} \geq 0,\ \ k = 1, 2, \ldots, n_c$$

Sheel and Scholtes 2000 describe in detail the connections.

Important concepts about MPCC and OMPV

- **Definition (ULSC).** A weakly stationary point (x, y, z, w) of (OMPV) satisfies the upper level strict complementarity (ULSC) property if there exists an MPCC multiplier that satisfies

$$y_k + w_k = 0 \;\Rightarrow \widehat{\eta}_{y,k}\widehat{\eta}_{w,k} \neq 0,\; k = 1, 2, \ldots, n_c.$$

- **Definition (MPCC-LICQ)** MPCC-LICQ holds at a feasible (x, y, z, w), point of (OMPV) if the gradients of all active constraints of (OMPV) at (x, y, z, w), with the exception of the complementary constraint $y^T w \leq 0$, are linearly independent.

Note (Sheel and Scholtes 2000) Under MPCC-LICQ, all stationarity concepts are the same at a solution point of (OMPV).

Assumptions

A1 The mappings f, g, h, F are twice continuously differentiable.

A2 The constraints involving only the parameters x satisfy, for any x,

- **i)** $\nabla_x h(x)$ has full column rank.
- **ii)** $\exists p \in R^n$ such that $\nabla_x h(x)p = 0$ and $\nabla g_i(x)p < 0$ whenever $g_i(x) \geq 0$.
- **iii)** The linearization $h(x) + \nabla_x h(x)d = 0$, $g(x) + \nabla_x g(x)d \leq 0$ is feasible.

A3 The partition $[\nabla_y F, \nabla_w F, \nabla_z F]$ is a mixed P partition (3).

Assumptions about the algorithm

Definition (Global Convergence Safeguard). A nonlinear programming algorithm (such as **FilterSQP**) whose outcome is

1. An infeasible point of the nonlinear program at which the linearization of the constraints is infeasible.
2. A feasible point of the nonlinear program that does not satisfy MFCQ.
3. A feasible point of the nonlinear program that satisfies MFCQ and that is a KKT point of the nonlinear program.

Alg1 The nonlinear programming algorithm has a global convergence safeguard.

Then any accumulation point of a nonlinear programming algorithm that satisfies Assumption **Alg1** and is applied to (OMPV(c)) is a KKT point!

ε stationary point, dual conditions

$(x, y, w, z, \zeta_1, \zeta_2)$ is an ε stationary point of (OMPV(c)) if there exists $(\lambda, \mu, \theta, \eta_y, \eta_w, \alpha_c, \alpha_1, \alpha_2)$ such that:

$$\left\{\begin{array}{rcl} \nabla_x f(x,y,w,z)^T + \nabla_x h(x)^T \lambda + & & \\ \nabla_x g(x)^T \mu + \nabla_x F(x,y,w,z)^T(\theta^+ - \theta^-) & = & t_x \\ \nabla_y f(x,y,w,z)^T + \eta_y + \alpha_c w + \nabla_y F(x,y,w,z)^T(\theta^+ - \theta^-) & = & t_y \\ \nabla_w f(x,y,w,z)^T + \eta_w + \alpha_c y + \nabla_w F(x,y,w,z)^T(\theta^+ - \theta^-) & = & t_w \\ \nabla_z f(x,y,w,z)^T + \nabla_z F(x,y,w,z)^T(\theta^+ - \theta^-) & = & t_z \\ \|\theta^+\|_1 + \|\theta^-\|_1 + \alpha_1 = c + t_{\alpha 1};\ \alpha_c + \alpha_2 = c + t_{\alpha 2} & & \\ \mu \geq \mathbf{0};\ \eta_{\mathbf{y}}, \eta_{\mathbf{w}} \geq \mathbf{0}; \theta^+, \theta^- \geq \mathbf{0};\ \alpha_{\mathbf{c}}, \alpha_{\mathbf{1}}, \alpha_{\mathbf{2}} \geq \mathbf{0}, & & \end{array}\right.$$

$$\|t_x, t_y, t_w, t_z, t_{\alpha 1} t_{\alpha 2}\|_\infty \leq \varepsilon.$$

ε stationary point, primal and compl. conditions

$$\left\{\begin{array}{rcl} g(x) & \le & t_g \\ h(x) & = & t_h \\ -\zeta_1 e_{n_c+l} - t_{1F} & \le & F(x,y,w,z) \le \zeta_1 e_{n_c+l} + t_{2F} \\ \mathbf{y}, \mathbf{w} & \le & \mathbf{0} \\ y^T w & \le & \zeta_2 + t_c \\ \boldsymbol{\zeta_1}, \boldsymbol{\zeta_2} & \ge & \mathbf{0}, \end{array}\right.$$

$$\left\{\begin{array}{l} (\zeta_1 e_{n_c+l} - F)^T \theta^+ + (F + \zeta_1 e_{n_c+l})^T \theta^- = t_{cF} \\ \alpha_c(\zeta_2 - w^T y) = t_{cc};\ g(x)^T \mu = t_{cg}; \\ |\alpha_2 \zeta_2| \le t_{cp};\ |\alpha_1 \zeta_1| \le t_{cp};\ \left|y^T \eta_y\right| \le t_{cp};\ \left|w^T \eta_w\right| \le t_{cp}, \end{array}\right.$$

$$\left\| t_g, t_h, t_{1F}, t_{2F}, t_c, t_{cc}, t_{cF}, t_{cg}, t_{cp} \right\|_\infty \le \varepsilon.$$

... piece of cake for interior-point methods

The algorithm

Choose $c_0 > 0$, $n = 0$, $K > 1$, an integer $q \geq 1$ and a sequence $\varepsilon^n \to 0$.

MPCC Find an ε^n solution $(x^n, y^n, w^n, z^n, \zeta_1^n, \zeta_2^n)$ of (OMPV(c^n)).

If $\zeta_1^{c^n} + \zeta_2^{c^n} > (\varepsilon^n)^{\frac{1}{q}}$,

update c : $c^{n+1} = Kc^n$ and n : $n = n + 1$.

return to **MPCC**

Note that, as opposed to **Scholtes 2002**, we do not need an infinite number of steps to solve the subproblem.

Global Convergence Theorem

Assume that

- (OMPV) satisfies the assumptions **A1**, **A2** and **A3**.
- (OMPV(c^n)) is solved with an NLP algorithm that satisfies Assumption **Alg1** that produces an ε^n stationary point.
- $\lim_{n\to\infty} c^n \varepsilon^n = 0$.
- The sequence $\left(x^{c^n}, y^{c^n}, w^{c^n}, z^{c^n}, \zeta_1^{c^n}, \zeta_2^{c^n}\right)$ has an accumulation point.

Then (1) if the penalty parameter update rule is activated a **finite** number of times any accumulation point is a **strongly stationary** point of (OMPV) and (2) if the penalty parameter update rule is activated an **infinite** number of times, and then any accumulation point is a **C-stationary** point of (OMPV).

Note that we may still diverge to ∞ ... but we'll fix that.

Approximate second-order stationary points

Definition (ϵ, χ second-order stationary point). We say that the point $\tilde{x} = (x, y, z, w, \zeta_1, \zeta_2)$, together with a Lagrange multiplier $\widetilde{\lambda} = (\lambda,\ \mu,\ \theta^{+n},\ \theta^{-n},\ \eta_y\ ,\eta_w\ ,\alpha_c\ ,\alpha_1\ ,\alpha_2)$ is an ε, χ second-order point of (OMPV(c)) if

1. $\tilde{x} = (x, y, z, w, \zeta_1, \zeta_2)$, is an ε stationary point of (OMPV(c)), that satisfies *exactly* the primal-dual complementarity involving the slack variables $\eta_{y,k}^T y = 0$, $\eta_{w,k}^T w = 0$.
2. $u^T \Lambda_{xx}^c(\tilde{x}, \tilde{\lambda}) u > 0$ for any u that is at the same time in the null space of the gradients of the active bound constraints of (OMPV(c)) and null space of a subset of the χ-active non-bound constraints of (OMPV(c)).

Note that sufficient conditions can be tested by by active set methods with rank-revealing factorization.

M-stationarity Result

Assume that

- The problem (OMPV) satisfies assumptions **A1**, **A2** and **A3**
- (OMPV(c^n)) is solved with an algorithm that satisfies Assumption **Alg1**.
- $\tilde{x}^n = (x^n, y^n, z^n, w^n, \zeta_1^n, \zeta_2^n)$ is a ε^n, χ^n second-order stationary point of (OMPV(c^n)), for all $n = 1, 2, \ldots, \infty$
- $\lim_{n\to\infty} c^n = \infty$, $\lim_{n\to\infty} \varepsilon^n = 0$, $\lim_{n\to\infty} \chi^n = 0$ and $\lim_{n\to\infty} c^n \varepsilon^n = 0$.
- $(x^*, y^*, z^*, w^*, \zeta_1^*, \zeta_2^*)$ is an accumulation point of this sequence.
- If (x^*, y^*, z^*, w^*) satisfies MPCC-LICQ,

then (x^*, y^*, z^*, w^*) must be an **M-stationary point of (OMPV)**.

Convergence to strongly stationary points

If, in addition to the assumptions of M-stationarity convergence we have that ULSC holds at the accumulation point (x^*, y^*, z^*, w^*), then (x^*, y^*, z^*, w^*) is a **strongly stationary** point and, as a result, a **strongly stationary** point.

The result is similar to the results of **Fukushima and Pang 98** and **Scholtes 2002**, except that it works with approximate points. **Sven's objection** However, if ULSC does not hold a descent direction may still exist.

Is M-stationarity sufficient?

Assume that (x^*, y^*, z^*, w^*) is an M-stationary point of (OMPV). Then, for any $\delta > 0$, the following exist

1. A perturbation $f^\delta(x, y, w, z)$of the objective function $f(x, y, w, z)$that satisfies $\left\|\nabla_{\tilde{x}} f^\delta(x, y, w, z) - \nabla_{\tilde{x}} f(x, y, z, w)\right\| \leq \delta$ for all $\tilde{x} = (x, y, z, w)$ in a neighborhood of (x^*, y^*, z^*, w^*).
2. A vector l_F^δ that satisfies $\left\|l_F^\delta\right\| \leq \delta$.
3. A point $(x^\delta, y^\delta, z^\delta, w^\delta)$ that satisfies $\left\|(x^\delta, y^\delta, z^\delta, w^\delta) - (x^*, y^*, z^*, w^*)\right\| \leq \delta$ and that is a **strongly stationary point (thus a B-stationary point)** for the perturbed problem

The perturbed problem

$$
(\delta OMPV) \qquad
\begin{array}{lll}
\min\limits_{x,y,w,z} & f^{\delta}(x,y,w,z) & \\
\text{sbj.to} & g(x) & \leq 0 \\
 & h(x) & = 0 \\
 & F(x,y,w,z) & = l_F^{\delta} \\
 & y, w & \leq 0 \\
 & (y^T w = 0) \;\; y^T w & \leq 0
\end{array}
$$

M-stationary points my be indistinguishable in finite arithmetic or for finite tolerance from strongly-stationary points!

Finishing global convergence: keep iterates finite

A4 The penalty function $\psi(x, y, w, z) = ||F(x, y, w, z)||_\infty + y^T w$ has bounded level sets over the set defined by the constraints $g(x) \leq 0,\ h(x) = 0,\ y \leq 0,\ w \leq 0.$

A5 The objective function $f(x, y, w, z)$ is bounded below over the same set.

Alg2 For any fixed value of c, the algorithm that is applied for solving the problem (OMPV(c)) decreases the merit function $f(x, y, z, w) + c\psi(x, y, z, w)$.

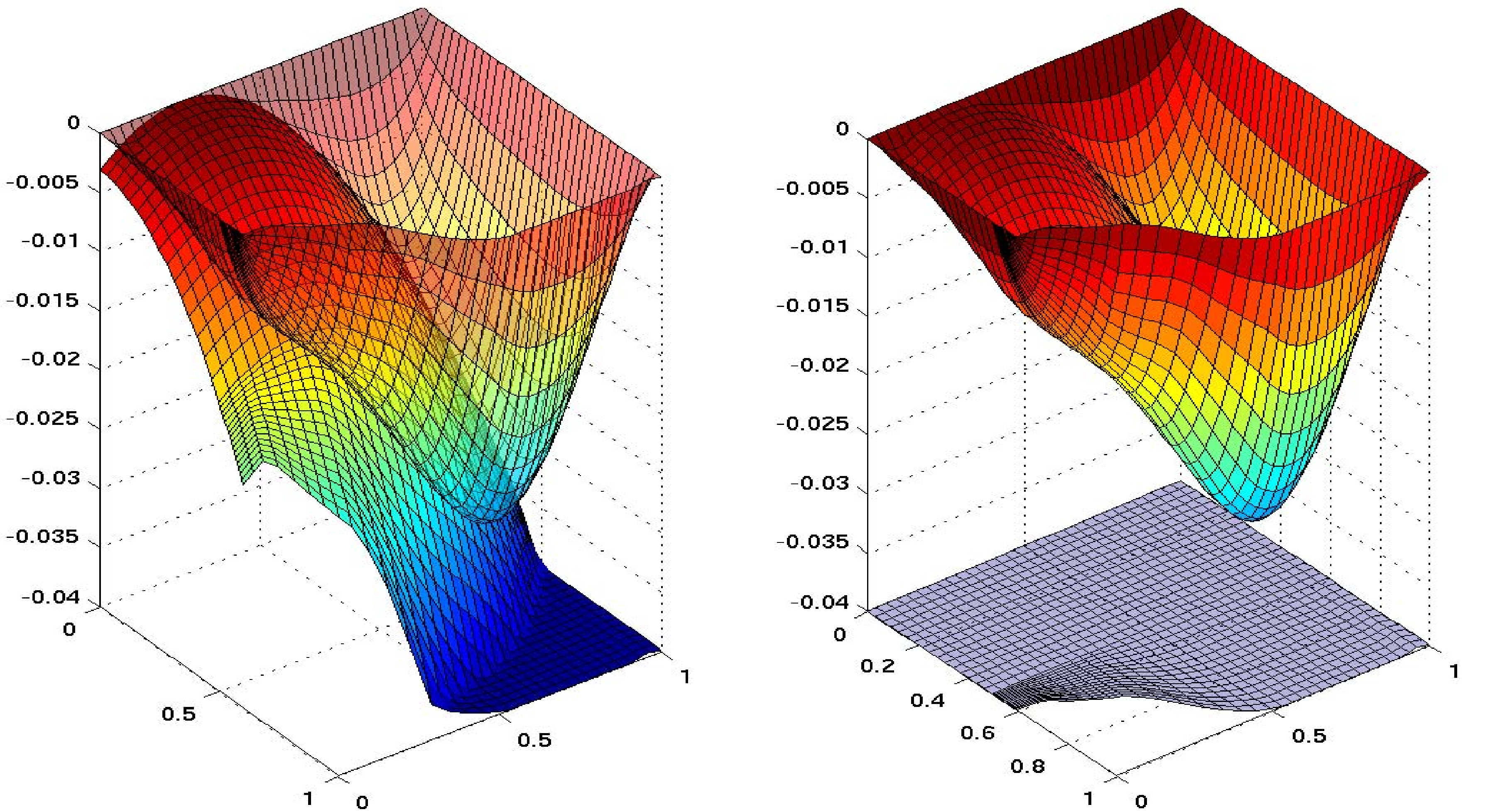
The merit function $\Psi(x, y, w, z, c) = \frac{1}{c}\left(f(x, y, w, z) - B_f\right) + \psi(x, y, w, z)$ **is always decreasing (even at penalty update) and has bounded level sets** $\Rightarrow$ convergence to C-stationary points is guaranteed!

The obstacle problem

$$
\text{(OBST)} \quad
\begin{array}{lll}
\displaystyle\min_{x,y,w,z} & f(x,z) & \\
\text{sbj.to} & g(x) & \le 0 \\
& -A(x)z + \phi(x) & = y \\
& k(\phi(x) - A(x)z) + \chi(x) - z & = w \\
& y, w & \le 0 \\
& (y^T w = 0) \;\; y^T w & \le 0
\end{array}
$$

We proved that the obstacle problem satisfies assumptions A1, A2, A3, A4 !!! So not so outlandish after all.

A graph of the obstacle problem



Packaging with rigid parabolic obstacle.

The obstacle problem test set (THANKS SVEN!!)

All of them satisfy Assumption **A5**

- **The incidence set identification problem** The contact region must be as close as possible to a prescribed shape.
- **The packaging problem with compliant obstacle**. Minimize the area of the membrane, while keeping the membrane in contact with the obstacle over at least a prescribed region.
- **The packaging problem with rigid obstacle**. Same as before but the obstacle is rigid.

Algorithmic choices for our numerical simulations

1. We use `knitro` to solve OMPV(c), the relaxed problem. `knitro` was not proven to satisfy **Alg1**, but we can test for ϵ stationarity and `knitro` provided one for any problem.

2. $q = 2$, $K = 10$, $c_0 = 10$, and $\epsilon^n = 10^{-3}12^{-n}$. We put ϵ^n=`opttol=feastol`.

3. Stopping Criteria $\zeta_1^n + \zeta_2^n \leq 1e - 7$.

4. **Note that $c^n \leq 10^{n+1}$, means that $c^n \epsilon^n \to 0$, as required by our results!!**

Detecting C-stationarity and M-stationarity

- We construct what we hope are good MPEC multipliers:

$$\widehat{\eta}_{w,k} = \eta_{w,k} + c y_k, \quad \widehat{\eta}_{y,k} = \eta_{y,k} + c w_k, \quad k = 1, 2, \ldots, n_C.$$

- We define

$$\text{Cstat} = \min_{k=1,2,\ldots,n_C} \widehat{\eta}_{w,k} \widehat{\eta}_{y,k}, \quad \text{Mstat} = \max_{k=1,2,\ldots,n_C} \min\{\widehat{\eta}_{w,k}, \widehat{\eta}_{y,k}\}.$$

- If Cstat ≥ 0 we go to a C-stationary point; if $Mstat \leq 0$, we have also an M-stationary point **(Note that the MacMPEC library uses nonnegativity constraints, as opposed to nonpositivity as used here)**.

Numerical Results

Problem	Obj	Uc	Ut	Cstat	Mstat	Feval	KFeval
is-1-8	2.352e-08	0	5	4.10e-11	2.89e-09	204	390
is-1-16	8.639e-06	1	6	9.38e-08	7.85e-06	451	4001
is-1-32	5.904e-06	2	7	3.36e-08	5.52e-05	2906	1097
is-2-8	4.517e-03	1	6	5.12e-08	2.84e-07	302	1712
is-2-16	3.006e-03	1	6	1.27e-06	1.02e-03	434	4001
is-2-32	1.774e-03	2	5	1.01e-05	3.54e-03	2083	4001
pc-1-8	6.000e-01	1	5	6.32e-14	1.40e-03	75	4001
pc-1-16	6.169e-01	1	7	3.82e-21	5.65e-07	297	4001
pc-1-32	6.529e-01	2	6	9.60e-18	8.93e-05	4999	3081
pc-2-8	6.731e-01	1	5	1.01e-19	3.03e-06	78	1421
pc-2-16	7.271e-01	2	5	3.60e-16	1.77e-03	289	1358
pc-2-32	7.826e-01	2	6	1.84e-16	1.22e-04	645	1350
pr-1-8	7.879e-01	1	6	9.28e-18	1.03e-06	193	81
pr-1-16	8.260e-01	2	5	1.68e-16	1.14e-05	218	54
pr-1-32	8.508e-01	2	5	1.95e-17	1.17e-03	644	3040
pr-2-8	7.804e-01	1	6	3.20e-18	1.46e-06	183	33
pr-2-16	1.085e+00	3	6	2.32e-15	1.73e-05	342	208
pr-2-32	1.135e+00	3	6	1.36e-14	1.59e-04	661	2792

Note C-stationarity always satisfied, M-stationarity almost true.

M-stationary points under finite tolerance

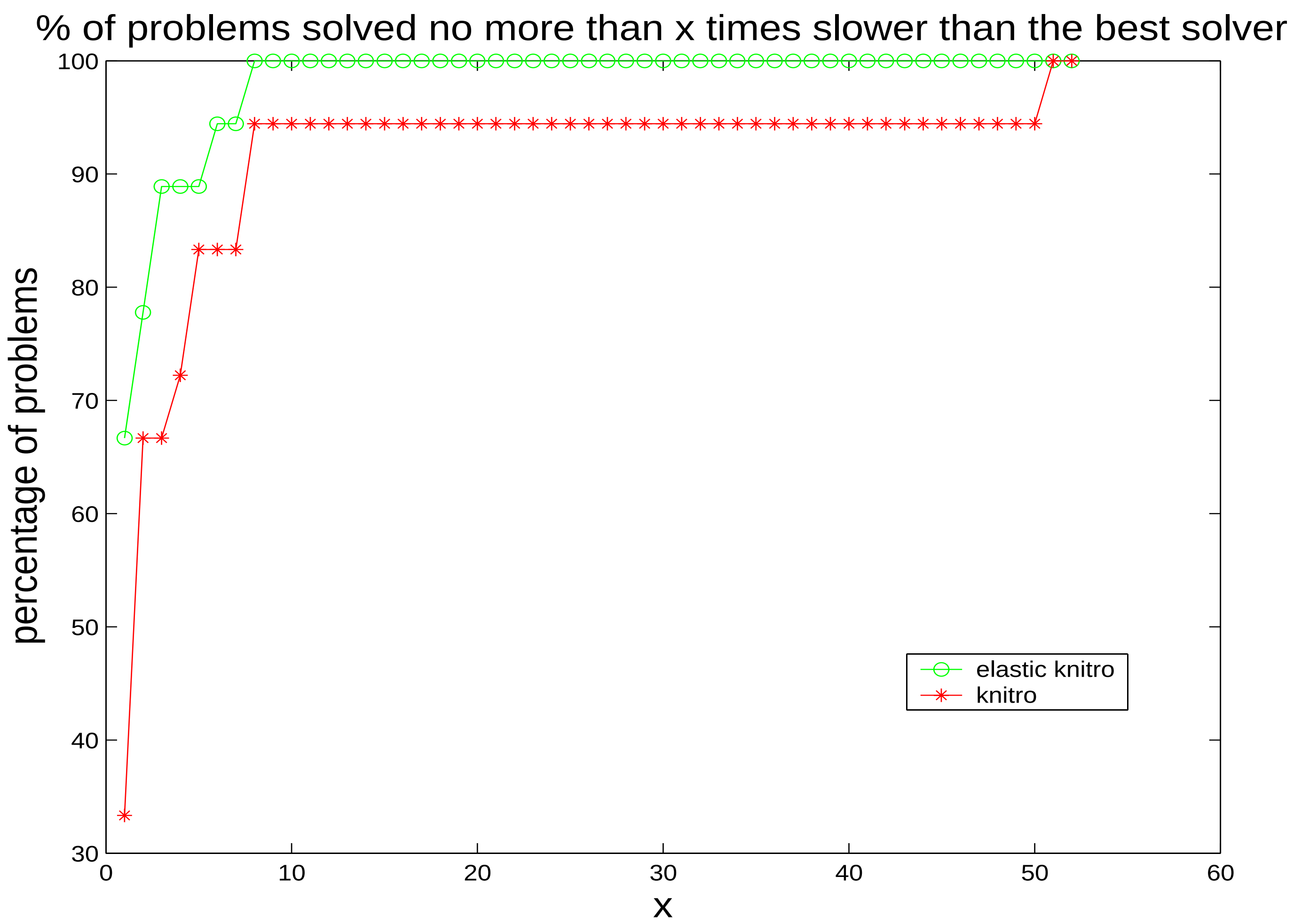
- The problem `pr-1-32`, for index $k = 19$ we have

$$y_{19} = 1.039e-05,\ w_{19} = 1.42e-04,\ \widehat{\eta}_{y,19} = 0.14,\ \widehat{\eta}_{w,19} = 1.17e-03$$

- In absence of any additional information (such as whether MPCC-LICQ holds, which cannot be tested for AMPL), it is difficult to decide whether the algorithm converges to an M-stationary point at which descent is still possible, or whether it converges to a strongly stationary point.

- However, if MPCC-LICQ holds, then one should somehow take advantage of **Sven's point**. But how to do that **before convergence**, is not clear.

The performance plot



Conclusions

- We proved that an elastic mode approach are guaranteed to converge to C-stationary points of the optimization of mixed P variational inequalities. **To my knowledge, the first that does not assume any other constraint qualification at the solution**.
- We proved that several variants of the obstacle problem satisfy our convergence assumptions.
- We have shown that M-stationary points **can be** confounded with strongly stationary points in finite arithmetic. This **does not mean that they will be** but in some of our examples they were.
- We have shown that our elastic mode approach with `knitro` solving the relaxed problem is superior to `knitro` alone at solving the problem.

Still to do

- Can one robustly marry this approach with an active-set approach to take advantage of MPCC-LICQ (if it holds) sufficiently close to the solution?
- Can convergence to M-stationarity hold under weaker conditions? For example MPCC-MFCQ (see Jane Ye's talk from Sunday)?