Using Linear Complementarity Techniques to Model and Simulate Multi-Rigid-Body Dynamics with Contact and Friction

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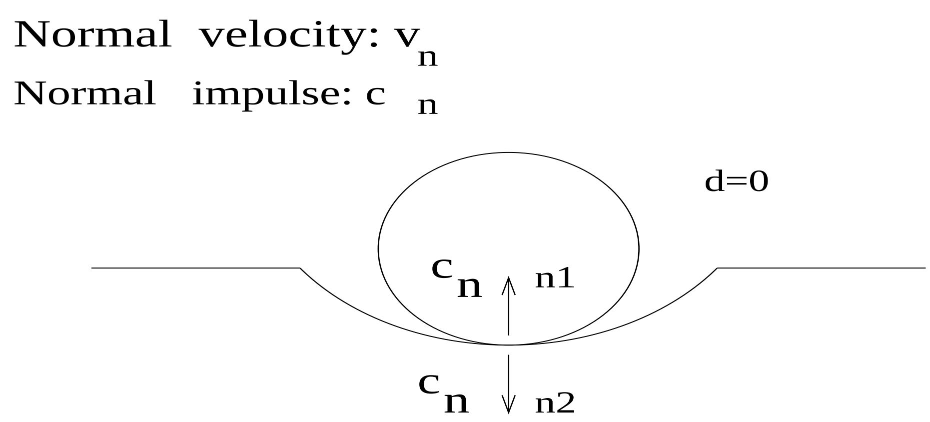
Model Requirements and Notations

- MBD system : generalized positions q and velocities v. Dynamic parameters: mass $M(q)$ (positive definite), external force $k(t, q, v)$.
- Non interpenetration constraints: $\Phi^{(j)}(q) \geq 0, \ 1 \leq j \leq n_{total}$ and compressive contact forces at a contact.
- Joint (bilateral) constraints: $\Theta^{(i)}(q) = 0, \ 1 \leq i \leq m$.
- Frictional Constraints: Coulomb friction, for friction coefficients $\mu^{(j)}$.
- Dynamical Constraints: Newton laws, conservation of impulse at collision.



Contact Model

- Contact configuration described by the (generalized) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.
- Contact forces are compressive, $c_n \geq 0$.
- Contact forces act only when the contact constraint is exactly satisfied, or

$$\Phi(q) \text{ is complementary to } c_n \quad \text{or } \Phi(q)c_n = 0, \quad \text{or } \Phi(q) \perp c_n.$$

Coulomb Friction Model

- Tangent space generators: $\widehat{D}(q) = \left[\widehat{d}_1(q), \widehat{d}_2(q)\right]$, tangent force multipliers: $\beta \in R^2$, tangent force $D(q)\beta$.
- Conic constraints: $||\beta|| \le \mu c_n$, where μ is the friction coefficient.
- Max Dissipation Constraints: $\beta = \text{argmin}_{||\widehat{\beta}|| \le \mu c_n} v^T \widehat{D}(q)\widehat{\beta}$.
- v_T, the tangential velocity, satisfies $|v_T| = \lambda = -v^T \widehat{D}(q) \frac{\beta}{||\beta||}$. λ is the Lagrange multiplier of the conic constraint.
- Discretized Constraints: The set $\widehat{D}(q)\beta$ where $||\beta|| \le \mu c_n$ is approximated by a polygonal convex subset: $D(q)\tilde{\beta}$, $\tilde{\beta} \ge 0$, $\left|\left|\tilde{\beta}\right|\right|_1 \le \mu c_n$. Here $D(q) = [d_1(q), d_2(q), \ldots, d_m(q)]$.

For simplicity, we denote $\tilde{\beta}$ the vector of force multipliers by β.

Acceleration Formulation

$$M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^{p} \left(n^{(j)}(q) c_n^{(j)} + D^{(j)}(q) \beta^{(j)} \right) = k(t, q, \frac{dq}{dt})$$

$$\Theta^{(i)}(q) = 0, \qquad i = 1 \ldots m$$

$$\Phi^{(j)}(q) \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j = 1 \ldots p$$

$$\beta = \text{argmin}_{\widehat{\beta}^{(j)}} v^T D(q)^{(j)} \widehat{\beta}^{(j)} \quad \textbf{subject to} \quad \left|\left|\widehat{\beta}^{(j)}\right|\right| \leq \mu^{(j)} c_n^{(j)}, \; j = 1 \ldots p$$

Here $\nu^{(\mathbf{i})} = \nabla \mathbf{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$.

It is known that these problems do not have a classical solution even in 2 dimensions, where the discretized cone coincides with the total cone.

Time-stepping scheme

Use Euler method, half-explicit in velocities, linearizing the geometrical constraints. Fundamental variables: velocities and impulses (h $\times$ force).

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)^T} v^{l+1} = 0, \qquad i = 1..m$$

$$\rho^{(j)} = n^{(j)^T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Here $\nu^{(\mathbf{i})} = \nabla \boldsymbol{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$. h is the time step. The set $\mathcal{A}$ consists of the active constraints. Stewart and Trinkle, 1996, MA and Potra,1997: The time-stepping scheme has a solution although the classical formulation doesn't!

Matrix Form of the Integration Step

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} -Mv^{(l)} - hk \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

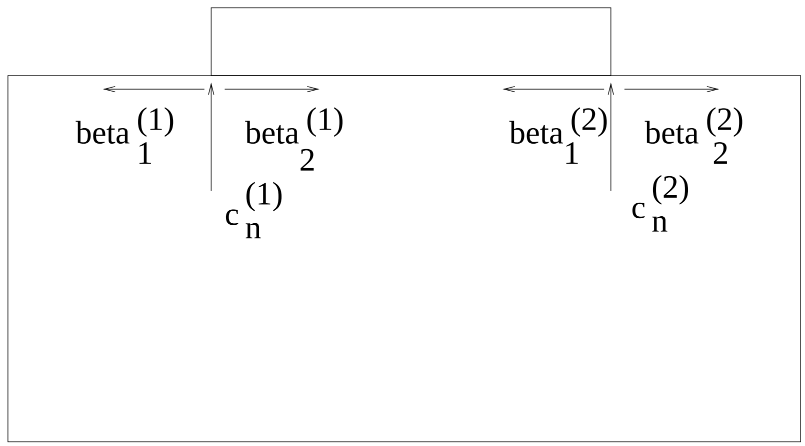
$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Solving the LCP

Is it possible to obtain an algorithm that has modest conceptual complexity?

- Lemke's method after reduction to proper LCP works, but for larger scale problems alternatives to it are desirable. Works well for tens of bodies, most of the time.
- Interior Point methods work for the frictionless problem (since matrices are PSD), but their applicability to the problem with friction depends on the convexity of the solution set.
- Iterative methods with convex subproblems–Fixed Point Iterations are unlikely to work in the case where the friction coefficient is large but many interesting configurations exhibit relatively small sliding friction coefficients.

Nonconvex solution set



1. $c_n^{(1)} = c_n^{(2)} = \frac{mhg}{2}$, $\beta_1^{(1)} = \beta_2^{(1)} = \beta_1^{(2)} = \beta_2^{(2)} = 0$, $\lambda^{(1)} = \lambda^{(2)} = 1$.
2. $c_n^{(1)} = c_n^{(2)} = \frac{mhg}{2}$, $\beta_1^{(1)} = \beta_2^{(2)} = 0$, $\beta_1^{(2)} = \beta_2^{(1)} = \mu\frac{mhg}{2}$, $\lambda^{(1)} = \lambda^{(2)} = 0$.

The average of these solutions, that both induce $v = 0$, violates,

$$\beta_1^{(2)} \geq 0 \quad \perp \quad \lambda^{(2)} \geq 0.$$

Efficiency of IP may be problematic, for any $\mu > 0$ (no $P*$ matrix)

$P_1(\Gamma)$: the first relaxation

Define $\Theta^{(l)} = -Mv^{(l)} - hk^{(l)}$. Let $P_1(\Gamma)$ be the convex mixed LCP:

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\ \tilde{\nu}^T & 0 & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 & -\tilde{\mu} \\ \tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\ 0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} + \begin{bmatrix} \Theta^{(l)} \\ 0 \\ \Gamma \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \\ \tilde{\lambda} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \\ \tilde{\zeta} \end{bmatrix} \geq 0.$$

Note that a solution of $P_1(\Gamma)$ with $\Gamma = \tilde{\mu}\tilde{\lambda}$, is a solution of the original LCP.

$P_2(\tilde{\lambda})$: the second relaxation

Let $P_2(\tilde{\lambda})$ be the convex mixed LCP:

$$\begin{bmatrix} M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} \\ \tilde{\nu}^T & 0 & 0 & 0 \\ \tilde{n}^T & 0 & 0 & 0 \\ \tilde{D}^T & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ \tilde{c}_\nu \\ \tilde{c}_n \\ \tilde{\beta} \end{bmatrix} + \begin{bmatrix} \Theta^{(l)} \\ 0 \\ 0 \\ E\tilde{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\sigma} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \end{bmatrix}^T \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{c}_n \\ \tilde{\beta} \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \tilde{\rho} \\ \tilde{\sigma} \end{bmatrix} \geq 0.$$

If $\tilde{\lambda}$ is the last component of a solution of the original LCP, then the velocity component of $P_2(\tilde{\lambda})$ is the same as the one in the original LCP. The solution dissipates and the noninterpenetration constraints are exactly satisfied.

Regularization of $\tilde{\lambda}$

For give velocity v define $\tilde{\lambda}$ in the following manner: for a contact j in the active contacts set $j \in \mathcal{A}$, define

$$\tilde{\lambda}^{(j)} = \max_{i=1,2,\ldots,m_C^{(j)}} \left\{ {d_i^{(j)}}^T v \right\}$$

This defines a map $v \to \tilde{\lambda}$ that we denote by $\Lambda(v)$. In some sense, this is the minimal $\tilde{\lambda}$ that is a part of the solution of the original LCP whose first component is v.

The nonconvexity of the solution in the previous example can be interpreted in the ambiguity of the choice of a component of $\tilde{\lambda}$ corresponding to zero tangential velocity at the contact.

Geometrical Regularity

We assume that the geometrical constraints satisfy the Mangasarian-Fromovitz condition, or in dual form

$$\tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n = 0, \quad \tilde{c}_n \geq 0 \Rightarrow \tilde{c}_\nu = 0, \;\; \tilde{c}_n = 0.$$

In the absence of an external force and friction, the internal forces are nonzero.

In primal form, this is equivalent to $\tilde{\nu}$ has full rank and there exist an u such that

$$\tilde{\nu}^T u = 0, \quad \tilde{n}^T u > 0.$$

This is equivalent to saying that the structure can be disassembled at zero initial conditions with external action.

Note that this assumption is weaker than linear independence of the constraints!

Fixed point iterations and related algorithms

1. Start: $\Gamma = 0$. Repeat until convergence: $v = P_1(\Gamma)$, $\tilde{\lambda} = \Lambda(v)$, $\Gamma = \tilde{\mu}\Gamma$.

2. Start: $\Gamma = 0$. Repeat until convergence: $v = P_1(\Gamma)$, $\tilde{\lambda} = \Lambda(v)$, $v = P_2(\tilde{\lambda})$, $\tilde{\lambda} = \Lambda(v)$, $\Gamma = \tilde{\mu}\Gamma$. This algorithm has the advantage that even if it does not converge and it is stopped, it will produce a partial solution that satisfies the noninterpenetration constraints exactly, exhibits a constrained force that belongs to the cover of the normal and tangent spaces (though it may not necessarily be in the friction cone) and dissipates energy.

3. Use algorithm 2 for exactly one iteration (same observation).

Algorithms 1 and 2 are fixed point iterations that are based on the fact that any solution of the original nonconvex LCP is a fixed point the composite maps.

Convergence results for fixed point iterations

Assume that the configuration is regular (disassemblable).

- There exist μ^* depending on the problem data at step l , such that whenever $||\tilde{\mu}|| \leq \mu^*$, the original nonconvex problem has a unique velocity solution to which both the first and the second algorithm will converge.
- Note that these results are different from other fixed point iterations in that we do not assume linear independence of normal and/or tangent vectors (even before discretization). In effect, we allow for the situation, very common in 3D, where there may be more than one constraint force (impulse) solution. The key issue here is that our iteration is velocity based and not force based.

Algorithm3

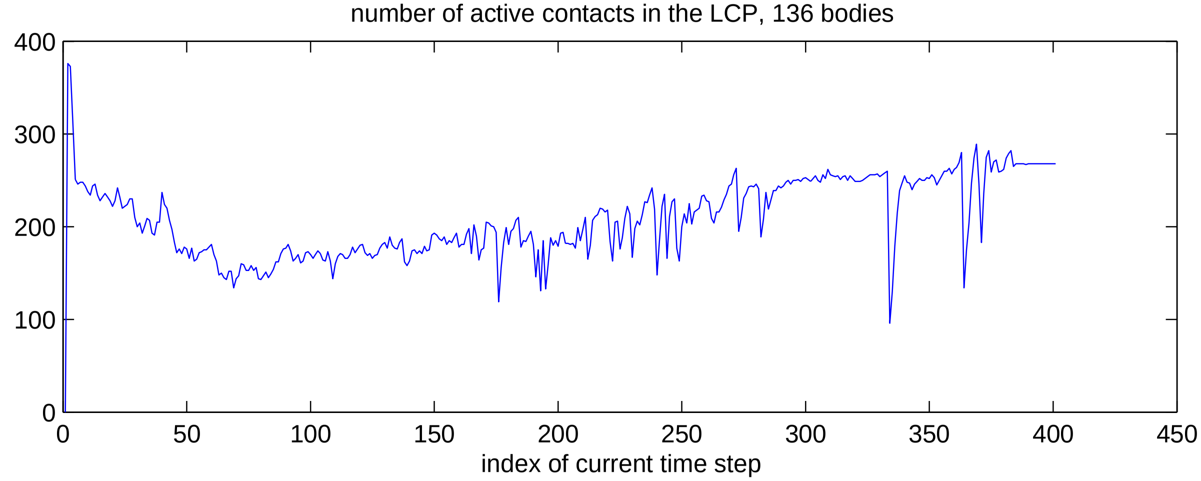
- Whenever μ belongs to set over which the friction cone is uniformly pointed, there exists an $L > 0$ dependent on the data of the problem such that, if $v, \tilde{\lambda}$ are components of the solution of the original LCP we have that

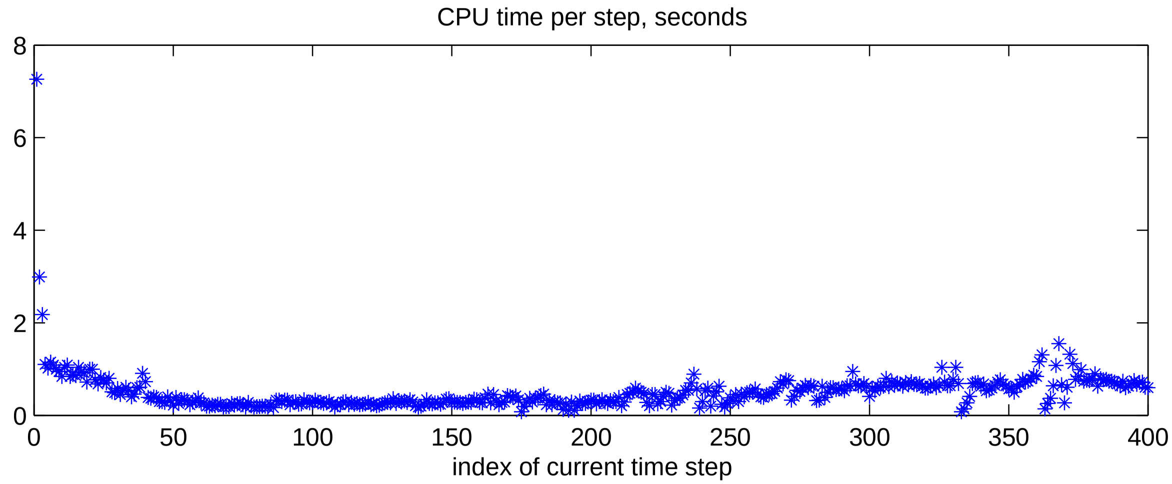
$$\left|\left|v - v^3\right|\right| \leq L \left|\left|\tilde{\mu}\tilde{\lambda}\right|\right|$$

  where v^3 is the outcome of the third algorithm.

- Algorithm 3 constitutes a relatively cheap and very good approximation when either we have small friction or large friction but small tangential velocity. In particular, if not contacts are in motion then the approximation is exact $v = v^3$. Unfortunately, once the solution to problem 3 is obtained, the distance to the solution set of the original LCP is hard to evaluate.

Results of a 2D simulation with 136 bodies





Algorithm 3. On the nonconvex formulation, PATH fails to find solution.

Conclusions

- We have constructed two algorithms with convex subproblems that are guaranteed to converge linearly to the unique velocity solution of the LCP of a time-stepping scheme for multibody dynamics with contact and friction whenever the friction coefficient is sufficiently small and the configuration is regular.
- Our regularity assumption is weaker than linear independence of the constraints involved and our framework can accomodate even some problems with nonconvex velocity-impulse solution sets.
- For a small number of bodies and contacts the method is not competitive with nonconvex LCP solvers. But solvers specific for convex quadratic programs are yet to be tested.
- We show that an efficient approximation can be constructed for cases that exhibit either small friction coefficients or small tangential velocity at the contact.