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On the Dynamic Stability of Electricity Markets

Victor M. Zavala · Mihai Anitescu

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Abstract In this work, we present new insights into the dynamic stability of electricity markets. In particular, we discuss how short forecast horizons, incomplete gaming, and physical ramping constraints can give rise to stability issues. Using basic concepts of market efficiency, Lyapunov stability, and predictive control, we construct a new stabilizing market design. A numerical case study is used to illustrate the developments.

Keywords Dynamics · Markets · Electricity · Efficiency · Game-Theory · Stability

1 Introduction

Electricity market models have become an indispensable tool for analyzing and predicting the impact of diverse dynamic drivers (e.g., weather, load, fuel prices, and wind supply), physical constraints (e.g., ramping, transmission congestion), and gaming behaviors (e.g., bidding strategies) on market efficiency and prices [26]. These models range from data-based time-series models [23,8] to mechanistic models based on agent-based systems [7,25] and game-theoretical formulations [6,15]. Game-theoretical models, in particular, make a systematic use of mechanistic gaming insights and physical constraints and thus provide more comprehensive predictive capabilities.

Several game-theoretical models based on a range of market structure assumptions have been proposed in the past. Most of these models are *static* in the sense that they assume some sort of steady-state behavior (e.g., periodicity) of the dynamic drivers. In the absence of dynamic constraints, these models can provide a reasonable representation of the long-term behavior of a market. Consequently, these models are useful

V. M. Zavala
Mathematics and Computer Science Division, Argonne National Laboratory
9700 South Cass Ave. Argonne, IL 60439 USA
Tel.: +1-630-252-3343
Fax: +1-630-252-5676
E-mail: vzavala@mcs.anl.gov

M. Anitescu
E-mail: anitescu@mcs.anl.gov

in analyzing steady-state constraints such as transmission congestion in planning and market design exercises. Static models, however, cannot explain the coupled effect of dynamic constraints and non stationary behavior of dynamic drivers on future price stability, which is critical in day-ahead and real-time operations.

A widely used game-theoretical dynamic market model was originally proposed in [1,2]. This model assumes that the players bid recursively in time in the direction that minimizes their marginal cost. Every bidding step can be interpreted as a steepest-descent step that converges to a steady-state equilibrium as time evolves. While this model is useful for analyzing market stability properties, it is based on mathematical rather than mechanistic assumptions and thus has limited applicability. Recently, a dynamic market model based on predictive control concepts was proposed in [14,13]. Here, supply functions and receding horizon concepts are incorporated in the model, providing a more natural representation of actual bidding procedures. This model has been used to analyze the effect of dynamic disturbances such as wind on prices under high penetration levels. A limitation of this framework, however, is that the dynamic model of the players is still based on the marginal-cost descent assumption.

Dynamic market models based on mechanistic bidding and physical constraints considerations have also been proposed [9,19,17]. These models can be used to explain how fluctuations of fundamental variables such as prices can arise from physical dynamic constraints such as ramping. Ramp constraints depend on multiple physical factors such as generator controller performance [3,4], thermal stresses, and wall capacitances [24]. These dynamic constraints affect market performance in a similar way as transmission congestion does [12].

The effect of dynamic constraints on market stability will become stronger in the presence of more volatile environments, such as those expected under high wind-power supply and smart-grid programs. Motivated by this situation, we revisit some of the issues affecting market stability. In particular, we establish a control-theoretical framework that uses concepts arising in electricity markets, dynamic games, and Lyapunov stability of predictive control. We derive a market-specific Lyapunov function that can be used to study the long-term stability of a given market. The Lyapunov function is constructed by using a summarizing state in terms of a basic definition of market efficiency. We demonstrate that the framework can be used to design, implement, and analyze market clearing procedures and gaming rules. In particular, we use the framework to explain how incomplete gaming solutions (as those used in practice), short foresight horizons, and limited ramping capacity can significantly affect market stability.

The paper is structured as follows. In Section 2 we present the market structure under consideration. In Section 3 we discuss implementation issues arising from incomplete gaming. In Section 4 we derive a framework to analyze market stability properties. In Section 5 we present a numerical case study. In Section 6 we provide concluding remarks and recommendations for future extensions.

2 Market Structure

We first define the market structure under consideration and discuss the underlying modeling assumptions.

2.1 Suppliers

We consider a supply-function equilibrium market structure similar to those proposed in [16,17]. Here, the supplier decisions are the parameters a_t^i, b_t^i of the affine supply function:

$$q_t^i(p_t, b_t^i, a_t^i) = b_t^i \cdot (p_t - a_t^i). \tag{1}$$

Here, q_t^i is the production quantity of supplier $i \in \mathcal{S} := \{1..S\}$ at time t; $p_t \geq 0$ is the price at time t, and a_t^i, b_t^i are the bidding coefficients at time t for supplier i. We assume that the supply function is non decreasing in p_t. Consequently, we impose the requirement that $b_t^i \geq 0$. In our analysis, we will assume that the generation quantities q_t^i and p_t are always non-negative. Consequently, will restrict the intercept parameter a_t^i to be non-negative as well. The supply function can also be expressed in inverse form as

$$p_t(q_t^i, b_t^i, a_t^i) = \frac{1}{b_t^i} q_t^i + a_t^i. \tag{2}$$

The consumer demands can also be represented as affine functions of the form

$$d_t^j = n_t^j - \gamma_t^j p_t, \tag{3}$$

where n_t^j is the nominal (inelastic) demand trajectory for consumer $j \in \mathcal{C} := \{1..C\}$ at time t and γ_t^j is the elasticity for consumer j. The inverse form of the demand function is

$$p_t(d_t^j, n_t^j, \gamma_t^j) = \frac{1}{\gamma_t^j}(n_t^j - d_t^j). \tag{4}$$

The supplier problem can be posed as follows. Starting at time k, given the price signals p_t over the future horizon $\mathcal{T} = \{k..k+T\}$, where T is the horizon length and $\hat{q}_k^i, \hat{a}_k^i, \hat{b}_k^i$ are the current states at time k for the supplier, find the bidding parameters trajectories $a_t^i, b_t^i,\ t \in \mathcal{T}$, that maximize the future profit (revenue minus marginal cost). To do so, we assume that the suppliers $i \in \mathcal{S}$ solve the following problem:

$$\max_{a_t^i, b_t^i, q_t^i} \sum_{t \in \mathcal{T}} \phi_t^i := \sum_{t \in \mathcal{T}} \left(p_t \cdot q_t^i - c_t^i(q_t^i) \right) \tag{5a}$$

$$\text{s.t. } q_t^i = b_t^i \cdot (p_t - a_t^i),\ t \in \mathcal{T} \tag{5b}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ t \in \mathcal{T} \tag{5c}$$

$$a_t^i, b_t^i \geq 0,\ t \in \mathcal{T} \tag{5d}$$

$$q_k^i = \hat{q}_k^i,\ a_k^i = \hat{a}_k^i,\ b_k^i = \hat{b}_k^i, \tag{5e}$$

where $\underline{q}^i, \overline{q}^i \geq 0$ are the lower and upper production limits, respectively. We emphasize that the quantities q_t^i in (5) act only as *dummy* variables. The accumulated future profit is denoted by $\sum_{t \in \mathcal{T}} \phi_t^i$. The marginal cost function is assumed to have the form

$$c_t^i(q_t^i) = h_t^i \cdot q_t^i + \frac{1}{2} g_t^i \cdot (q_t^i)^2. \tag{6}$$

We make the common assumption that $g_t^i > 0$ so the marginal cost is convex in q_t^i [21]. Consequently, we have that the supplier problem is convex in the space of q_t^i (e.g., the Cournot game case). However, we can observe that the problem is ill-posed in the space of a_t^i, b_t^i, since different combinations of these parameters can reach the same optimal quantities and profit. Since this introduces difficulties in analyzing the properties of the supplier problem, we will assume that the intercept parameters a_t^i are zero. This assumption will not affect the solution of the game as long as the price is assumed to be positive since there will always exist $b_t^i \geq 0$ that can map a given price to any production quantity satisfying (5c).

We also note that the case where $p_t = 0$ has a feasible solution only if $b_t^i = q_t^i = 0$ is admissible (i.e., the minimum capacity must be $\underline{q}^i = 0$). This can be seen from the optimality condition (20c) in the Appendix. We summarize the problem properties in the following statement.

Property 1 If $p_t \geq 0$, $a_t^i = 0$, and $g_t^i \geq 0$, problem (5) is convex. If $p_t > 0$, the problem has a feasible solution for any $\underline{q}^i, \overline{q}^i \geq 0$. If $p_t = 0$, the problem admits a solution only if $\underline{q}_t^i = 0$.

Since the quantities q_t^i in (5) act only as dummy variables, we can pose this problem entirely in terms of the prices p_t and the supply function parameters a_t^i, b_t^i by substituting (5b) into (5a) and (5c). In addition, we interpret the bidding parameters a_t^i, b_t^i as suppliers states. These modifications lead to the following equivalent formulation in state-space form:

$$\max_{b_t^i, \Delta b_t^i} \sum_{t \in \mathcal{T}} \left(p_t \cdot b_t^i \cdot p_t - c_t^i \left(b_t^i \cdot p_t \right) \right) \tag{7a}$$

$$\text{s.t. } b_{t+1}^i = b_t^i + \Delta b_t^i, \ t \in \mathcal{T}^- \tag{7b}$$

$$\underline{q}^i \leq b_t^i \cdot p_t \leq \overline{q}^i, \ t \in \mathcal{T} \tag{7c}$$

$$b_t^i \geq 0, \ t \in \mathcal{T} \tag{7d}$$

$$b_k^i = \hat{b}_k^i, \tag{7e}$$

where $\mathcal{T}^- := \mathcal{T} \setminus \{T\}$. The bidding increments Δb_t^i are interpreted as the control actions of the supplier. Note that these are unconstrained, implying that the suppliers can adjust their bids infinitely fast. A direct consequence is that the feasible set of the problem is invariant to the initial states $\hat{b}_k^i$. In addition, the feasible set is invariant to the price signals p_t since it is always possible to find $b_t^i \geq 0$ mapping any p_t to a feasible quantity q_t^i. Consequently, we denote the feasible set of this problem as Ω^i.

To simplify our analysis, we assume that the demands are inelastic. This is equivalent to assuming that the consumers do not bid into the market and $d_t^j = n_t^j$. This assumption will not affect the generality of our framework. However, it is well known that, in practice, elasticity can have significant effects on market stability.

2.2 ISO Market Clearing

The independent system operator (ISO) receives the bidding states b_t^i and clears the market by determining the generation quantities (and implicitly the prices) that balance total supply and demand. The main objectives of the ISO are to maximize social welfare and efficiency and to ensure market stability. The interaction between the ISO and the suppliers results in a game in which each player tries to maximize its own performance matric. The interaction is sketched in Figure 1.

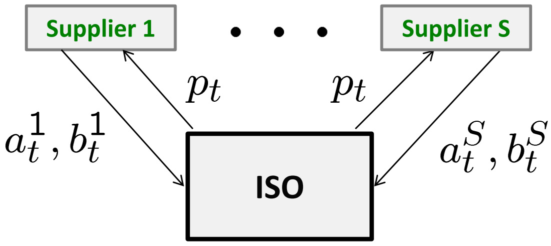

Fig. 1 Schematic representation of game between ISO and suppliers.

In this work, market stability will be interpreted as the ability to keep prices bounded from a given reference in the presence of dynamic fluctuations of demands and renewable supply and physical constraints. To do so, we propose to use the basic concept of *market efficiency* as a measure of stability. To define efficiency, we first define an *ideal unconstrained* market clearing problem. This problem can be stated as follows. Given supply function states b_t^i, solve [6]:

$$\min_{q_t^i} \quad \sum_{t\in\mathcal{T}} \bar{\varphi}_t := \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{S}} \int_0^{q_t^i} p_t(q, b_t^i)dq \tag{8a}$$

s.t.

$$\sum_{i\in\mathcal{S}} q_t^i \geq \sum_{j\in\mathcal{C}} d_j^t, \; t\in\mathcal{T} \tag{8b}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i, \; i\in\mathcal{S}, t\in\mathcal{T}, \tag{8c}$$

where

$$\int_0^{q_t^i} p_t(q, b_t^i)dq = \frac{1}{2\, b_t^i}(q_t^i)^2. \tag{9}$$

The objective function is the *negative social welfare*, denoted as $\sum_{t\in\mathcal{T}} \bar{\varphi}_t$. Since we have assumed that the consumers do not bid into the market, this reduces to the aggregated income of the suppliers. We have that $\bar{\varphi}_t \geq 0$ since $q_t^i, b_t^i \geq 0$. The multipliers for the constraint (8b) are the prices $\bar{p}_t \geq 0$. Note that the feasible set of this problem is not affected by the bidding parameters, since they enter only in the objective function. In addition, in this unconstrained formulation, we assume that the generators can move infinitely fast between production levels (no ramp constraints). This assumption decouples the problem in time. Hence, the feasible set of this problem is invariant to the current state of the generators $\hat{q}_k^i$.

The unconstrained market clearing problem is convex for fixed $b_t^i \geq 0$. The case where $b_t^i = 0$ only has a feasible solution if $q_t^i = 0$ is admissible (i.e., $\underline{q}^i = 0$). This can be seen from the optimality condition (24b) in the Appendix. In this case, it is possible to eliminate q_t^i from the formulation by fixing its value to zero. The problem always has a feasible solution as long as the demand is reachable. This can be achieved if the demand satisfies $\sum_{i\in\mathcal{S}} \underline{q}^i \leq \sum_{j\in\mathcal{C}} d_j^t \leq \sum_{i\in\mathcal{S}} \overline{q}^i,\ t \in \mathcal{T}$. We summarize these properties in the following.

Property 2 If $b_t^i \geq 0$, problem (8) is convex. The problem has a feasible solution if $\sum_{i\in\mathcal{S}} \underline{q}^i \leq \sum_{j\in\mathcal{C}} d_j^t \leq \sum_{i\in\mathcal{S}} \overline{q}^i$ holds. If $b_t^i > 0$, feasibility holds for any $\underline{q}_t^i, \overline{q}_t^i \geq 0$. If $b_t^i = 0$, the problem admits a solution only if $\underline{q}_t^i = 0$.

For our analysis, we note that having infinitely fast dynamics in the generators is equivalent to assume that their ramp capacities are equal to the distance between the maximum and minimum generation capacities $\overline{q}^i$ and $\underline{q}^i$, respectively. Thus, we can pose (8) in the following equivalent state-space form:

$$
\begin{aligned}
\min_{q_t^i, \Delta q_t^i} \quad & \sum_{t\in\mathcal{T}} \bar{\varphi}_t := \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{S}} \int_0^{q_t^i} p_t(q, b_t^i) dq && \text{(10a)} \\
\text{s.t.} \quad & \\
& q_{t+1}^i = q_t^i + \Delta q_t^i,\ i \in \mathcal{S},\ t \in \mathcal{T} && \text{(10b)} \\
& \sum_{i\in\mathcal{S}} q_t^i \geq \sum_{j\in\mathcal{C}} d_j^t,\ t \in \mathcal{T} && \text{(10c)} \\
& -(\overline{q}^i - \underline{q}^i) \leq \Delta q_t^i \leq (\overline{q}^i - \underline{q}^i),\ i \in \mathcal{S}, t \in \mathcal{T}^- && \text{(10d)} \\
& \underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i \in \mathcal{S}, t \in \mathcal{T} && \text{(10e)} \\
& q_k^i = \hat{q}_k^i,\ i \in \mathcal{S}. && \text{(10f)}
\end{aligned}
$$

The variables Δq_t^i are the generation ramp increments that are bounded by $\pm(\overline{q}^i - \underline{q}^i)$, the maximum generation ramp that is physically possible. We will see in the following proposition that it is possible to drop the dynamic constraints (10b). Hence, the feasible set invariant to the initial states of the suppliers $\hat{q}_k^i$. Accordingly, the feasible set of this problem will be denoted as $\Omega_{UNC}^{ISO}(\hat{q}_k^i)$ or Ω_{UNC}^{ISO}.

Proposition 1 *Problems* (8) *and* (10) *are equivalent.*

Proof: The unconstrained problem (8) generates optimal trajectories $\{q_t^i\},\ i \in \mathcal{S}$. Since $\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ t \in \mathcal{T}$, we have $-(\overline{q}^i - \underline{q}^i) \leq q_{t+1}^i - q_t^i \leq (\overline{q}^i - \underline{q}^i), t \in \mathcal{T}^-$. Moreover, this trajectory is invariant to the initial states $\hat{q}_k^i$ since $\underline{q}^i \leq \hat{q}_k^i \leq \overline{q}^i$. For problem (10), since the ramp increments Δq_t^i are bounded by $\pm(\overline{q}^i - \underline{q}^i)$, the optimal trajectories of (8) can be reached from any initial condition $\hat{q}_i^k$. This is equivalent to removing the variables Δq_t^i, dynamic constraints (10b), and initial conditions (10f). □

The solution of the unconstrained market clearing problem represents the *ideal* performance for the market (in the absence of ramping constraints). We now consider the *constrained* market clearing problem:

$$\min_{q_t^i,\Delta q_t^i} \sum_{t\in\mathcal{T}} \varphi_t := \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{S}} \int_0^{q_t^i} p_t(q,b_t^i)dq \tag{11a}$$

$$\text{s.t. } q_{t+1}^i = q_t^i + \Delta q_t^i,\ i\in\mathcal{S}, t\in\mathcal{T}^- \tag{11b}$$

$$\sum_{i\in\mathcal{S}} q_t^i \geq \sum_{j\in\mathcal{C}} d_j^t,\ t\in\mathcal{T} \tag{11c}$$

$$-\underline{r}^i \leq \Delta q_t^i \leq \overline{r}^i,\ i\in\mathcal{S}, t\in\mathcal{T}^- \tag{11d}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i\in\mathcal{S}, t\in\mathcal{T} \tag{11e}$$

$$q_k^i = \hat{q}_k^i,\ i\in\mathcal{S}. \tag{11f}$$

The multipliers for the constraint (11c) are the prices $p_t \geq 0$. In this formulation, the ramps are bounded by $\underline{r}^i, \overline{r}^i \leq (\overline{q}^i - \underline{q}^i)$, respectively. This constrains the dynamic response of the generators. As before, we note that the bidding parameters b_t^i enter only the cost function and thus do not affect the feasible set. In this case, however, the dynamic constraints introduce time coupling because the ramp constraints might become active. Consequently, the feasible set does depend on the initial conditions $\hat{q}_k^i$. Accordingly, the feasible set of this problem will be denoted as $\Omega^{ISO}(\hat{q}_k^i)$.

The constrained social welfare is denoted as $\sum_{t\in\mathcal{T}}\varphi_t$ with $\varphi_t \geq 0$ since $b_t^i, q_t^i \geq 0$. It is easy to prove that $\sum_{t\in\mathcal{T}}\varphi_t \geq \sum_{t\in\mathcal{T}}\bar{\varphi}_t$ since $\Omega^{ISO}(\hat{q}_k^i) \subseteq \Omega^{ISO}_{UNC}(\hat{q}_k^i)$. In other words, the performance of the constrained clearing problem is bounded by that of the unconstrained counterpart. It is not obvious, however, that $\varphi_t \geq \bar{\varphi}_t$ holds point wise since the constrained problem (11) exhibits time coupling. We prove this in a different way in the following proposition.

Proposition 2 *For fixed $b_t^i \geq 0$, the point social welfare φ_t evaluated at a solution of problem* (11) *and $\bar{\varphi}_t$ evaluated at a solution of* (8) *satisfy $\varphi_t \geq \bar{\varphi}_t$, $t\in\mathcal{T}$.*

Proof: At $t = k$ we have that $\varphi_k = \bar{\varphi}_k$ since the initial conditions $\hat{q}_k^i$ are fixed. The unconstrained cost $\bar{\varphi}_{k+1}$ is invariant to the state of the current time step since there are no ramp constraints. Consequently, there does not exist a feasible combination of increments Δq_k^i that can reach a feasible state q_{k+1}^i satisfying $\varphi_{k+1} < \bar{\varphi}_{k+1}$. Using induction over $t = k, ..., k+T$, we have that $\varphi_t \geq \bar{\varphi}_t$ point wise with equality if and only if the ramp constraints are non-binding. □

We now formally define the *point market efficiency* η_t as

$$\eta_t := \frac{\bar{\varphi}_t}{\varphi_t} = 1 - \frac{\varphi_t - \bar{\varphi}_t}{\varphi_t}, \quad t\in\mathcal{T}. \tag{12}$$

By definition and from Proposition 2, we have that $\eta_t \in [0,1]$. The case where $\eta_t = 1$ is achieved if and only if $\varphi_t = \bar{\varphi}_t$. This implies that the prices are close to those of the unconstrained market clearing problem, which represents the ideal market performance. The case where $\eta_t = 0$ occurs if and only if the constrained social welfare diverges to infinity. This case occurs when the future demands cannot be met given the current states the generators and the ramping constraints. This implies that the prices p_t diverge (i.e., a small change in demand leads to large changes in price). In the following section, we show that the distance in performance between the constrained and unconstrained games can be bounded by the magnitude of the ramp limits.

2.3 Stability of the Game Variational Inequality

Using the equivalence between (10) and (8), we have that the only difference between problems (10) and (11) are the ramp lower and upper bounds. Consequently, problem (11) results from parameter embedding of (10) with perturbations $\overline{r}^i - (\overline{q}^i - \underline{q}^i)$ and $\underline{r}^i - (\overline{q}^i - \underline{q}^i), i \in \mathcal{S}$. Using this observation we can analyze stability for the solution of the unconstrained game under perturbations of the ramp limits and thus relate its solution to that of the constrained game.

To establish stability of the unconstrained game, we need to ensure that the mapping matrix of the variational inequality resulting from coupling the optimality conditions of problems (10) and (8) is nonsingular. The optimality conditions of these problems are given in the Appendix.

Lemma 1 *Consider the following block matrix (where the diagonal blocks are square).*

$$L = \begin{bmatrix} G & 0 & 0 & A & 0 \\ I & P & 0 & BH & 0 \\ 0 & -B^{-2} & B^{-1} & M^T & N^T \\ 0 & 0 & M & 0 & 0 \\ 0 & 0 & N & 0 & 0 \end{bmatrix}$$

We make the following assumptions:

[A1] The matrix G is invertible.
[A2] The matrices P, B are diagonal and positive, with entries equal to the prices p_t and bidding parameters b_i^t, respectively.
[A3] The blocks G, P, B have the same dimensions.
[A4] The matrix $[M^T N^T]$ has full column rank.
[A5] The diagonal entries of B are bounded below.

Then, there exist positive values q_*, p_*, independent of G such that, if $p_t \cdot b_t^i > q_*$ and $p_t > p_*$, then the matrix L is nonsingular.

Proof: Since from assumptions [A1],[A2] we have that G, P are invertible, we immediately have the following algebraic relationship

$$\begin{bmatrix} G & 0 \\ I & P \end{bmatrix}^{-1} = \begin{bmatrix} G^{-1} & 0 \\ -P^{-1}G^{-1} & P^{-1} \end{bmatrix}$$

We construct the Schur complement of the upper 2×2 block of blocks, which we denote by S

$$S = \begin{bmatrix} B^{-1} & M^T & N^T \\ M & 0 & 0 \\ N & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -B^{-2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G^{-1} & 0 \\ -P^{-1}G^{-1} & P^{-1} \end{bmatrix} \begin{bmatrix} 0 & A & 0 \\ 0 & BH & 0 \end{bmatrix}$$
$$= \begin{bmatrix} B^{-1} & M^T & N^T \\ M & 0 & 0 \\ N & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \left[B^{-2}P^{-1}G^{-1}A - B^{-1}P^{-1}H\right] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} B^{-1} & \left[M^T - B^{-2}P^{-1}G^{-1}A + B^{-1}P^{-1}H\right] & N^T \\ M & 0 & 0 \\ N & 0 & 0 \end{bmatrix}$$

From the properties of Schur complements (and of determinants) it follows that L is invertible if and only if S is invertible. Using the Schur complement argument again, we obtain that since B (Assumption [A2]) is invertible, then S is invertible if and only if the Schur complement

$$T = \begin{bmatrix} M \\ N \end{bmatrix} B \begin{bmatrix} M \\ N \end{bmatrix}^T + \begin{bmatrix} M \\ N \end{bmatrix} \left[\left[-B^{-1}P^{-1}G^{-1}A + P^{-1}H \right] 0 \right]$$

is invertible. But from [A4], [A5] it follows that the first matrix in the sum has eigenvalues bounded below away from 0. Since the matrices M, N, G^{-1}, A, H are fixed it follows that if $P^{-1}B^{-1}$ and P^{-1} are sufficiently small, then the matrix T is positive definite (even if not symmetric) and thus invertible. The conclusion follows. $\square$.

We now analyze the stability of the solution of the game created by the coupled solution of (5) and (11).

Theorem 1 *Let J be the reduced Jacobian of the game* (5) *and* (11) *(the Jacobian of the coupled KKT conditions of the game, with the variables that reached their bounds eliminated). Then, if at a solution of the game each of the optimization problems satisfies LICQ and the prices p_t, $t \in \mathcal{T}$ and the production q_t^i, $t \in \mathcal{T}$, $i \in \mathcal{S}$ values are large enough, then J is invertible.*

Proof: We write the supplier optimization problem (5) in a slightly different way by extracting the bidding trajectory parameters b_t^i outside the problem (we are considering only the case with $a_t^i = 0$) and solving (5) in terms of q_t^i. Subsequently, b_t^i is obtained by solving for it from the supply function equation (5b). To make the intended application of Lemma 1 more clearly justified, we rewrite (5b) as $q = B * (I_{\mathcal{S}} \otimes I_{\mathcal{T}})\, p$. Here q are the overall production levels, B is a matrix whose diagonal are the bidding parameters, and p is the vector of prices. Here $\otimes$ is the Kronecker product.

Then b_t^i is used as a parameter in the optimization problem of the ISO (11). In the latter optimization problem, we eliminate Δq_t^i using (11b).

It then follows that the active Jacobian of the game that couples (5), (5b), and (11) has precisely the structure of the matrix from Lemma 1. The variables (corresponding to the columns of that matrix) are, in order, q_t^i, λ (production levels and Lagrange multipliers of the supplier problem), b_t^i, q_t^i (production levels of the ISO problem), the prices p_t, and the Lagrange multipliers of the ISO problem.

The first row corresponds to the gradient of the KKT conditions of (5). The matrix G is the KKT matrix of that problem reduced on the active set, and the matrix A appears from the gradient of the KKT conditions with respect to the price (which is the only exogenous variable that impacts the supplier optimization problem). The second row corresponds to the Jacobian of the equation (5b). Here we define $P = (I_{\mathcal{S}} \otimes I_{\mathcal{T}})\, p$. We also define H to be the matrix that satisfies $B\,(I_{\mathcal{S}} \otimes I_{\mathcal{T}})\, p = BHp$ (such a matrix must exist since the form on the left is bilinear in b and p). We note that (5b) can then also be written as $q = BH \cdot p$.

The last three rows correspond to the KKT conditions of the ISO game where the fourth row corresponds to the demand satisfaction constraint and the last row to all other constraints. Moreover, they are coupled with the other optimization problem only through the parameters B that account for the B^{-2} block in the (3,2) position.

We note that assumptions of Lemma 1 are satisfied. Indeed [A1] follows from the assumption that LICQ holds for (5) and the fact that its objective function is strongly convex in q. Assumptions [A2],[A3] follow by construction. Assumption [A4] follows from the fact that (11) satisfies LICQ.

We then obtain from Lemma 1 that the reduced Jacobian matrix is invertible as long as $p_t \cdot b_t^i$ and p_t are sufficiently large. From (5b) the first is equivalent with q_t^i being sufficiently large, which proves the claim. □

We note that our stability condition is necessary but not sufficient for the stability of the solution of the resulting variational inequality [11], at least not in the general case. The typical result for stability involves a P property, which we do not prove here. On the other hand, such a result is sufficient in the case where strict complementarity holds.

In any case, the result gives us an interesting insight, which is that among the cases for which we can guarantee stability are the cases where the prices and production levels are sufficiently high. This seems a reasonable conclusion from a modeling perspective. Assuming stability of the solution of the underlying variational inequality given by the game, we can establish the following result that bounds the distance between the solution of the unconstrained and constrained games

Theorem 2 *Assume that a solution of the unconstrained game $\bar{\varphi}_t, \bar{p}_t, \bar{\eta} = 1, t \in \mathcal{T}$ given by* (7) *and* (10) *is locally stable. Then, there exist Lipschitz constants $L_\varphi, L_p, L_\eta \geq 0$ such that the solution of the constrained game $\varphi_t, p_t, \eta_t,\ t \in \mathcal{T}$ given by* (7) *and* (11) *satisfies,*

$$
\begin{aligned}
|\varphi_t - \bar{\varphi}_t| &\leq L_\varphi \sum_{i \in \mathcal{S}} \left(|\overline{r}^i - (\overline{q}^i - \underline{q}^i)| + |\underline{r}^i - (\overline{q}^i - \underline{q}^i)| \right) \\
|p_t - \bar{p}_t| &\leq L_p \sum_{i \in \mathcal{S}} \left(|\overline{r}^i - (\overline{q}^i - \underline{q}^i)| + |\underline{r}^i - (\overline{q}^i - \underline{q}^i)| \right) \\
|\eta_t - 1| &\leq L_\eta L_\varphi \sum_{i \in \mathcal{S}} \left(|\overline{r}^i - (\overline{q}^i - \underline{q}^i)| + |\underline{r}^i - (\overline{q}^i - \underline{q}^i)| \right).
\end{aligned}
$$

Proof: The result is immediate from stability of the solution of the unconstrained game [11], from the fact that the constrained game is a parametric embedding of the unconstrained counterpart, and from the definition of efficiency (12). □

We note that the Lipschitz constants depend on the initial conditions of the generators, on the demand, and on the horizon length. In particular, if the demand and the initial conditions are such that the ramp constraints are not active, the Lipschitz constants will be zero.

In this analysis, we have not considered network constraints in order to simplify the presentation. However, the definition of efficiency can account for any other physical constraints limiting performance with respect to the unconstrained clearing problem (8).

Based on the definition of efficiency, maximizing efficiency is equivalent to minimizing the social welfare. While this is an obvious result from a modeling point of view, we will see that adding the efficiency definition in the market clearing problem (11) is advantageous from a market stability point of view.

3 Implementation Issues

To represent the game given by (7) and (11) in abstract form, we define the market states x_t as the set of quantities q_t^i and prices p_t and define the aggregated vector from time k to $k+T$ as $\mathbf{x}_k^{k+T} := \{x_k, ..., x_{k+T}\}$ with initial conditions $\hat{\mathbf{x}}_k$. The controls u_t are defined as the set of ramps for all suppliers $\Delta q_t^i, i \in \mathcal{S}$ with $\mathbf{u}_k^{k+T-1} = \{u_k, ..., u_{k+T-1}\}$. The bidding increments Δb_t^i are interpreted as the supplier controls and are denoted as w_t^i, and we define $w_t := \{w_t^1, ..., w_t^S\}$. We define the supplier vectors $\mathbf{w}_k^{k+T-1,i}, i \in \mathcal{S}$ and the total vector $\mathbf{w}_k^{k+T-1}$. The bidding states b_t^i are interpreted as the supplier states z_t with aggregated vector $\mathbf{z}_k^{k+T}$ and initial conditions $\hat{\mathbf{z}}_k$. We include the problem data over the horizon (in this case given by the demands) in the aggregated vector $\mathbf{m}_k^{k+T}$.

We can eliminate the states x_t, z_t by forward elimination. With this, we can express the supplier and market clearing problem entirely in terms of the controls and initial state conditions. We thus have the supplier problem,

$$\min_{\mathbf{w}_k^{k+T-1,i}} \sum_{t \in \mathcal{T}} \phi_t^i(w_t^i, u_t) \tag{13a}$$

$$\text{s.t. } \mathbf{w}_k^{k+T-1,i} \in \Omega^i, \tag{13b}$$

for $i \in \mathcal{S}$ and the constrained market clearing problem,

$$\min_{\mathbf{u}_k^{k+T-1}} \sum_{t \in \mathcal{T}} \varphi_t(u_t, w_t) \tag{14a}$$

$$\text{s.t. } \mathbf{u}_k^{k+T-1} \in \Omega^{ISO}(\hat{\mathbf{x}}_k). \tag{14b}$$

Since the decisions of the players do not affect each others feasible sets, the resulting game is a pure Nash equilibrium problem [10].

For implementation, the game given by (13) and (14) can be solved over a receding horizon. One way of doing so is as follows. At time k we use the forecast data $\mathbf{m}_k^{k+T}$

(e.g.; demands $d_t^j,\ t \in \mathcal{T} = \{k..k+T\}$) and the current states $\hat{\mathbf{x}}_k, \hat{\mathbf{z}}_k$. We solve the game (13) and (14) over the horizon $\mathcal{T}$ to obtain $\mathbf{u}_k^{k+T-1^*}, \mathbf{w}_k^{k+T-1^*}$. The system will evolve from its current state $\hat{\mathbf{x}}_k, \hat{\mathbf{z}}_k$ into the state $\hat{\mathbf{x}}_{k+1}, \hat{\mathbf{z}}_{k+1}$. In the nominal case (no forecast errors in the data $\mathbf{m}_k^{k+T}$), the state will evolve as predicted from the model. At the next step $k+1$, we introduce feedback in the market by shifting the horizon of the game to obtain $\mathcal{T} \leftarrow \{k+1..k+T+1\}$ and use the new state $\hat{\mathbf{x}}_{k+1}, \hat{\mathbf{z}}_{k+1}$ as initial conditions. The new data $\mathbf{m}_{k+1}^{k+1+T}$ is forecast and the game problem is solved to obtained the new decisions $\mathbf{u}_{k+1}^{k+T^*}, \mathbf{w}_{k+1}^{k+T^*}$. Note that even in the nominal case feedback is required because the horizon T is usually finite (at time k it is not possible to foresee demands beyond time $k+T$). This implementation framework is intuitive, but it is not used in practice, mainly because of constraints in information exchange and in decision times.

The current strategy used in practice is to solve the game by iterating between the suppliers and the ISO in a distributed manner [20, 5]. Here, each supplier guesses the ISO states (e.g. prices) or, implicitly, its decisions. This guess is denoted by $\mathbf{u}_k^{k+T-1^\ell}$, where ℓ is an iteration counter. The suppliers compute bidding parameters $\mathbf{w}_k^{k+T-1,\ell}$ by solving (13). These are sent to the ISO to solve the market clearing problem (14) to update the controls $\mathbf{u}_k^{k+T-1,\ell+1}$. This can be interpreted as a Jacobi-like iteration.

The Jacobi iterate $\mathbf{u}_k^{k+T-1,\ell+1}, \mathbf{w}_k^{k+T-1,\ell+1}$ is feasible but not optimal for the game. Feasibility follows since the suppliers decisions $\mathbf{w}_k^{k+T-1}$ do not enter the feasible set $\Omega^{ISO}(\cdot)$ and since the supplier problems always have a feasible solution for any decisions of the ISO $\mathbf{u}_k^{k+T-1}$. This suboptimal strategy is an incomplete gaming strategy between the suppliers and the ISO. A key observation is that *the resulting incomplete gaming error generated at each step* is propagated forward in time through the initial states $\hat{b}_k^i, \hat{q}_k^i$ and thus introduces additional dynamics into the market. As we discuss in the next section, this can lead to market stability issues. For instance, the suboptimal gaming solution obtained at time k might place the generators at a future state $k+1$ from which the future demands at times $k+1...k+1+T$ cannot be reached, thus making the game infeasible at $k+1$ infeasible.

4 Dynamic Stability Issues

Stability, in the context of wholesale electricity markets, reflects strong fluctuations and divergence of prices. Traditional control-theoretic stability analysis tools are not directly applicable in this context because the market is inherently dynamic and does not exhibit a natural equilibrium for the states. While it is possible to design market clearing procedures (these can be viewed as *market controllers*) that artificially introduce equilibria (i.e., by enforcing periodicity in some form), this strategy can constrain and degrade market performance. New stability analysis tools are thus needed to enable a systematic design, analysis, and implementation of *robust* and stabilizing market clearing procedures that can sustain market manipulation and strong dynamic variations of demands and renewable supply. In this section, we take a first step toward this goal by making use of a market-specific Lyapunov stability framework.

We can express the market efficiency as an implicit function of the states of the form $\eta_k(x_k, z_k)$ or η_k for short-hand notation. Here, we use the following definition of market stability.

Definition 1 The market system defined by the game (13) and (14) is said to be *stable* if, given $\eta_0 \in \Omega^\eta(\epsilon) := \{\eta \,|\, \eta \geq \epsilon\}$ with $\epsilon \in [0, 1]$, there exist feasible sequences u_k, w_k over $k = 0..\infty$ such that $\eta_k \in \Omega^\eta(\epsilon)$.

Here, ϵ is an *efficiency threshold value*. We note that efficiency is a state derived from the system physical states. This value implicitly sets a measure of stability for the prices. Here, we propose to measure price stability as the distance between the prices of the constrained and unconstrained market clearing problems $|p_t - \bar{p}_t|$. Having such a relative measure is important since high efficiencies do not necessarily imply large prices and viceversa.

We now define the *summarizing market state*:

$$\delta_{k+1} := (1 - (\eta_k(x_k, z_k) - \epsilon)) \cdot \delta_k, \ k = 0..\infty, \tag{15}$$

with initial conditions $\delta_0 \geq \alpha > 0$. Here, we can use $\delta_0 := (1 - (\eta_0(x_0, z_0) - \epsilon))\mu$ with $\mu > 0$ as long as $\eta_0(x_0, z_0) \geq \epsilon$. If $\eta_k(\cdot, \cdot) \geq \epsilon$, $k = 0..\infty$, then for any $\alpha > 0$ such that $\delta_0 \geq \alpha$ there exists $\kappa \geq 0$ such that $\delta_k \to \kappa$ for all $k = 0..\infty$. In other words, the summarizing market state has a stable origin. Stability of this origin implies market stability in the sense of Definition 1. On the other hand, if at any step we have $\eta_k(\cdot, \cdot) < \epsilon$, the summarizing market state will increase. Subsequent violations of the efficiency threshold will make the summarizing state diverge from the origin.

Using this basic set of definitions, we now illustrate how to establish sufficient stability conditions for a given market design. In addition, we demonstrate that the current market design given by the incomplete solution of the game (13) and (14) is not stabilizing.

We propose to extend the market clearing problem (14) by making use of the definition of the summarizing state as follows.

$$\min_{\mathbf{u}_k^{k+\mathcal{T}-1}} \sum_{t \in \mathcal{T}^{-1}} (\delta_{t+1} - \delta_t) \tag{16a}$$

$$\text{s.t. } \mathbf{u}_k^{k+T-1} \in \Omega^{ISO}(\hat{\mathbf{x}}_k) \tag{16b}$$

$$\delta_{t+1} = (1 - (\eta_t(x_t, z_t) - \epsilon)) \cdot \delta_t, \ t \in \mathcal{T}^- \tag{16c}$$

$$\eta_t(x_t, z_t) \geq \epsilon, \ t \in \mathcal{T} \tag{16d}$$

$$\delta_k = \hat{\delta}_k, \tag{16e}$$

The detailed formulation of this problem is presented in the Appendix. The objective function of this market clearing problem will be used as a *summarizing market function*, which we define formally as

$$V(\delta_k) := -\sum_{t \in \mathcal{T}^{-1}} (\delta_{t+1} - \delta_t) = (\delta_k - \delta_T). \tag{17}$$

The solution of the game (13) and (16) provides the feedback law $(u_k, w_k) = h(\delta_k)$. A crucial observation is that the summarizing market function can be used as a Lyapunov function that we can use to establish stability of the origin for the summarizing state δ_k. To prove this, we first make the following definition.

Definition 2 A function $V(\delta_k)$ is a Lyapunov function for system $\delta_{k+1} = f(\delta_k, h(\delta_k))$ if (1) it is positive definite: in a region Ω containing the origin if for $\delta_k \in \Omega$ we have $V(\delta_k) \geq 0$ for $\delta_k \geq 0$ for all k, and (2) it is non increasing: $\Delta V(\delta_k) \leq 0$, for all k.

We now establish stability following the traditional approach of using the cost function of the controller (in this case market clearing problem) as a Lyapunov function [18].

Theorem 3 *If the game given by* (13) *and* (16) *has a feasible solution, the summarizing cost function* (17) *with infinite horizon* $T = \infty$ *is positive definite and non increasing.*

Proof: From feasibility of (16d) we have that $-(\delta_{t+1} - \delta_t) \geq 0,\ t \in \mathcal{T}^-$ so $V(\delta_k) = \sum_{t\in\mathcal{T}^-} -(\delta_{t+1} - \delta_t) \geq 0$. Consequently, positive definiteness follows. To prove that the function is non increasing, we consider the cost function of two consecutive problems generating two trajectories $\delta_t^k,\ t \in \{k..k+T\}$ and $\delta_t^{k+1},\ t \in \{k+1..k+1+T\}$ with $T = \infty$, $\delta_k^k = \delta_k$ and $\delta_{k+1}^{k+1} = \delta_{k+1}$. We then have

$$\begin{aligned}
\Delta V(\delta_k) &= V(\delta_{k+1}) - V(\delta_k) \\
&= \sum_{t=k+1}^{\infty} (\delta_{t+1}^{k+1} - \delta_t^{k+1}) - \sum_{t=k}^{\infty} (\delta_{t+1}^{k} - \delta_t^{k}) \\
&= (\delta_{k+1} - \delta_\infty^{k+1}) - (\delta_k - \delta_\infty^k) \\
&\leq (\delta_{k+1} - \delta_k) \\
&= (1 - (\eta_k - \epsilon)) \cdot \delta_k - \delta_k \\
&= -(\eta_k - \epsilon) \cdot \delta_k \\
&\leq 0.
\end{aligned}$$

The proof is complete. □

With this, we have established that the decay of the summarizing function is a sufficient condition for market stability. We note that if at any point we have that $\eta_k < \epsilon$, then $\delta_{k+1} > \delta_k$, and the decay condition will not hold.

A crucial observation in our analysis is the need of the incorporation of the stabilizing constraint (16d). With this, the feasible set of the market clearing problem depends on the bidding states of the suppliers. A consequence is that the ISO and the suppliers might need to iterate several times (e.g., in a Jacobi manner) to be sure of obtaining a feasible solution to the game. Another consequence of this analysis is the fact that the existing market design where a single iterate is performed between the ISO and the suppliers *cannot be guaranteed to be stable* in the sense of Definition 1 since not every set of bidding parameters can be guaranteed to lead to a market clearing solution satisfying the stabilizing constraint. In other words, the current market design does not enable the ISO to correct the bidding quantities to stabilize the market. Hence, the market is more prone to be manipulated and destabilized by the suppliers if these do not have appropriate means to guess the ISO decisions (e.g., by price forecasting). Finding a feasible solution to the game (13) and (16) avoids these problems. Our construct provides a mechanism to design and analyze market designs with stability guarantees.

We highlight the fact that, in practice, it is not strictly necessary to solve problem (16) as long as the stabilizing constraint (16d) is satisfied by the given market design. Another issue arising in implementation is the fact that the market clearing problem is normally solved over a finite receding horizon $\mathcal{T} = \{k..k+T\}$. Hence, even if the infinite horizon game is feasible, the solution of the receding horizon game cannot be guaranteed to be feasible. Stability conditions can also be established in this case but they require the existence of a stable *terminal controller* able to stabilize the summarizing state beyond the current terminal time $k+T$ [18]. Constructing such a controller in a systematic manner remains an open research question. However, it is possible to establish stability conditions for finite horizon controllers. For two consecutive problems we have

$$\begin{aligned}
\Delta V(\delta_k) &= V(\delta_{k+1}) - V(\delta_k) \\
&= \sum_{t=k+1}^{k+T} (\delta_{t+1}^{k+1} - \delta_t^{k+1}) - \sum_{t=k}^{k+T-1} (\delta_{t+1}^{k} - \delta_t^{k}) \\
&= (\delta_{k+1} - \delta_{k+1+T}^{k+1}) - (\delta_k - \delta_{k+T}^{k}) \\
&= (\delta_{k+1} - \delta_k) + (\delta_{k+T}^{k} - \delta_{k+1+T}^{k+1}).
\end{aligned}$$

Consequently, as long as $(\delta_{k+1} - \delta_k) \leq 0$ (equivalent to $\eta_k \geq \epsilon$) and $(\delta_{k+T}^{k} - \delta_{k+1+T}^{k+1}) \leq -(\delta_{k+1} - \delta_k)$, stability will follow. In other words, these conditions guarantee that the controller is making progress toward the origin.

5 Numerical Case Study

In this section, we illustrate the effect of ramping constraints, foresight horizon, and incomplete gaming solutions on market stability and price dynamics. We consider a market system with three suppliers and one demand. One of the suppliers has fast dynamics (high ramping capacity) but high cost such as natural gas generators, the second one has slow dynamics but also low cost such as a coal generator, and the third one is used as a slack generator with infinite ramp limits (equal to generation capacity) and a large cost. This last supplier acts as a slack to avoid infeasibility. The nominal parameters used are $\underline{q} = [0,0,0]$, $\overline{q} = [50,70,120]$, $\underline{r} = -[5,10,120]$, $\overline{r} = [5,10,120]$, $h = [4,2,5]$, and $g = [2,1,5]$. We used $\hat{q}_0 = [0,40,40]$ as initial conditions. We consider the demand profile presented in Figure 2, which is obtained from a periodic signal perturbed with Gaussian noise. We set the market stability threshold to $\epsilon = 0.65$.

To illustrate the main developments of the paper, we consider three market implementations. The first one uses a foresight horizon of six hours and performs a single Jacobi-like iteration at each clearing time (incomplete gaming). This implementation is labeled as $(T = 6Jac)$ and represents current practice. The second implementation

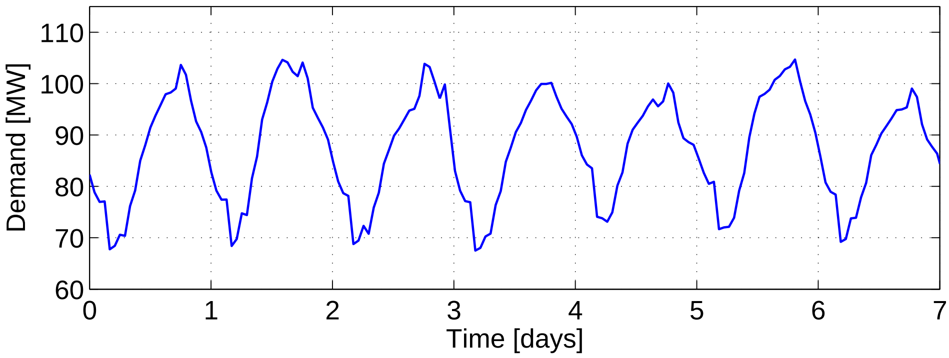

Fig. 2 Demand profile used for numerical case study.

uses the same horizon length, but the game is converged to optimality ($T = 6Opt$) satisfying the stabilizing constraint. The third implementation uses an horizon of 24 hours, and the game is converged to optimality ($T = 24Opt$). To compute the reference social welfare used in the definition of market efficiency, we also implemented an unconstrained market clearing procedure.

In Figure 3 we present the profiles of the summarizing state δ_t for the three market implementations, in Figure 4 we present efficiency profiles η_t, and in Figure 5 we present the resulting clearing price signals p_t. From Figure 3 it is clear that the summarizing state obtained from the suboptimal implementation $T = 6Jac$ is not strictly decreasing during days 1 and 3 and thus its market clearing cost cannot be used as a Lyapunov function. This indicates that the efficiency is crossing the threshold at certain times, as can be observed in Figure 4. This clearly illustrates that incomplete gaming can introduce market instability. The other two control implementations remain stable, but, as expected, a longer foresight horizon improves performance. This is observed from the faster decay of the summarizing state for $T = 24Opt$ when compared with $T = 6Opt$ and from the efficiency profiles. The efficiencies of $T = 24Opt$ remain farther away from the threshold. This illustrates that the length of the foresight horizon can have important effects on market stability. This is mainly because longer foresights can anticipate and manage ramping constraints more efficiently.

In Figure 5 we observe the spikes in the prices for $T = 6Jac$ during the first hours of the simulation and during the third day. In particular, note the strong price fluctuations when compared with the optimal unconstrained prices. These prices were obtained from the solution of the unconstrained market clearing problem. Note that in the absence of ramping constraints, the prices remain stable and nearly periodic. On the other hand, when the ramp constraints are active, strong price variations are observed. In particular, during the third day, the prices for $T = 6Jac$ reach levels of $150\$/MW$. The prices of $T = 24Opt$ stay well below $100\$/MW$ and much closer to the optimal unconstrained prices. These levels are a consequence of having a longer foresight horizon and converging the game to optimality to ensure that the efficiency is above the stability threshold. As a quantitative result, we computed the sum of squared errors $SSE = \sum_t |p_t - \bar{p}_t|^2$ over the entire simulation horizon of 7 days. Here, p_t are the constrained price signals, and $\bar{p}_t$ are the unconstrained price signals. For $T = 6Jac$ we obtained SSE=2.16×10^5 while for $T = 24Opt$ we have SSE=4.19×10^4. This is an improvement of nearly an order of magnitude. We have also observed that performing an extra Jacobi-like iteration for $T = 6Jac$ stabilizes the prices. In addition, we have observed that extending the horizon of $T = 24Opt$ does not improve its performance significantly.

In Figure 6 we present price profiles for $T = 6Jac$ and $T = 24Opt$ with relaxed ramp constraints. In this case, we increased the ramp limits from their nominal values to $\underline{r} = -[10, 20, 120]$, $\overline{r} = [10, 20, 120]$. As can be seen, the price signals for both implementations are close to those of the unconstrained clearing problem. The signals of $T = 24Opt$ get closer to the unconstrained reference faster because of a combined effect of complete gaming and forecast horizon. In particular, we observe that $T = 6Jac$ performs well in this case. The reason is that when the ramp limits are relaxed, subsequent gaming solutions become closer to each other. This case illustrates how ramping constraints can have strong effects on market efficiency and stability and how alternative market designs can help mitigate those effects.

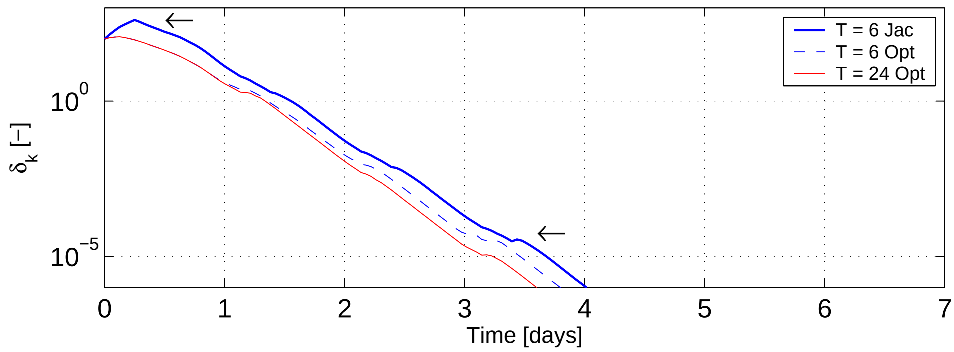

Fig. 3 Summarizing state for market implementations.

6 Conclusions and Future Work

We have established a framework to analyze and design stabilizing market designs. The framework uses a game theoretical framework incorporating physical constraints, market efficiency concepts, and Lyapunov analysis tools. We explain how market stability issues can arise in current market designs as a result of incomplete gaming between the ISO and the suppliers and short foresight horizons. The framework is general and can be extended and modified to consider other operational scenarios such as network constraints, forward and real-time markets, forecast errors, stochastic formulations, piece-wise supply functions, and Cournot games. In any of these developments, we believe it is critical to establish a consistent framework that can be used to design and compare different market designs by characterizing their stabilizing and robustness

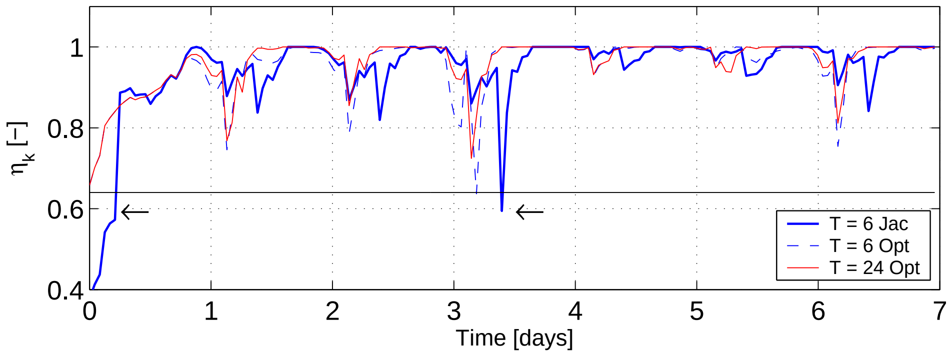

Fig. 4 Efficiencies for market implementations.

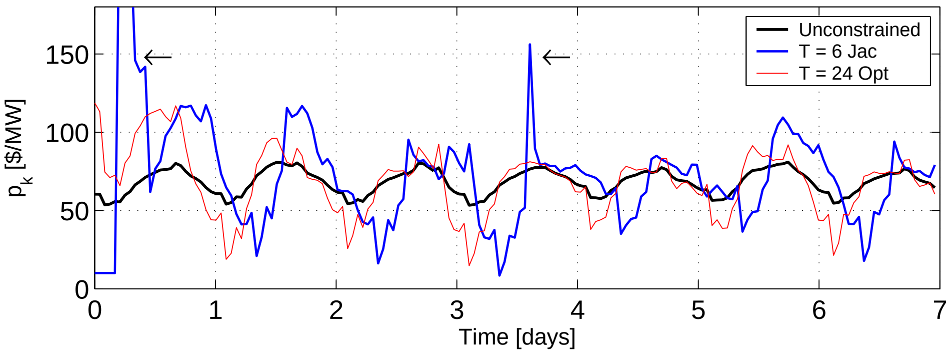

Fig. 5 Clearing prices for market implementations.

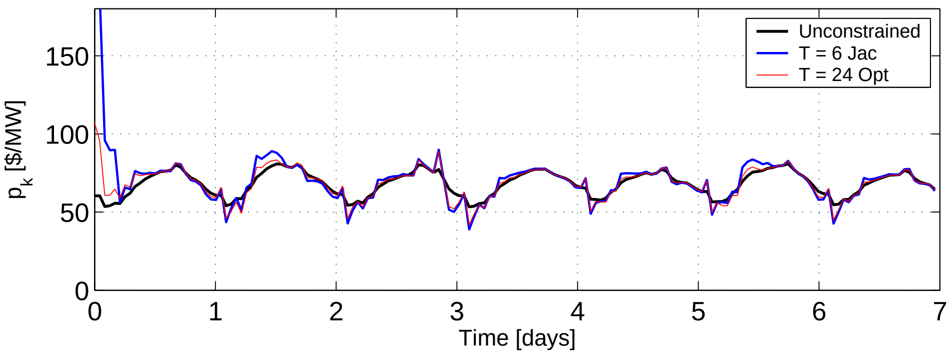

Fig. 6 Clearing prices for market implementations under relaxed ramp constraints.

properties. The issue of incomplete gaming opens the door to several questions regarding appropriate distributed approaches to implement the bidding-clearing procedure in real-time. In particular, Jacobi-like iterations cannot be guaranteed to converge [10]. A potential alternative would be to apply projected-gradient descent schemes [22].

Acknowledgments

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A Problem Formulations and Optimality Conditions

A.1 Suppliers

For the supplier problem (5) we have the following. Given the prices $p_t,\ t \in \mathcal{T}$ solve

$$\min_{b_t^i} \sum_{t\in\mathcal{T}} \left(c_t^i(q_t^i) - p_t \cdot q_t^i\right) \tag{18a}$$

$$\text{s.t. } q_t^i = b_t^i \cdot p_t,\ t \in \mathcal{T} \tag{18b}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ t \in \mathcal{T} \tag{18c}$$

$$b_t^i \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{18d}$$

$$q_k^i = \hat{q}_k^i,\ b_k^i = \hat{b}_k^i. \tag{18e}$$

Since this problem is decoupled in time, we can derive its optimality conditions by looking at the Lagrange function at a time instant t:

$$\begin{aligned}\mathcal{L}_t^i(p_t) = c_t^i(q_t^i) - p_t \cdot q_t^i + \lambda^{q}{}_t^i \cdot \left(q_t^i - b_t^i \cdot p_t\right)\\ - \underline{\nu}^{q}{}_t^i \cdot (q_t^i - \underline{q}_t^i) - \overline{\nu}^{q}{}_t^i \cdot (\overline{q}_t^i - q_t^i) - \nu^{b}{}_t^i \cdot b_t^i,\ i \in \mathcal{S}, t \in \mathcal{T}.\end{aligned} \tag{19}$$

The optimality conditions are

$$\nabla_{q_t^i}\mathcal{L}_t^i(p_t) = h_t^i + g_t^i \cdot q_t^i - p_t + \lambda^{q}{}_t^i - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{20a}$$

$$\nabla_{b_t^i}\mathcal{L}_t^i(p_t) = -\lambda^{q}{}_t^i \cdot p_t - \nu^{b}{}_t^i = 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{20b}$$

$$\nabla_{\lambda^{q}{}_t^i}\mathcal{L}_t^i(p_t) = q_t^i - b_t^i \cdot p_t = 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{20c}$$

$$0 \leq \underline{\nu}^{q}{}_t^i \perp (q_t^i - \underline{q}^i) \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{20d}$$

$$0 \leq \overline{\nu}^{q}{}_t^i \perp (\overline{q}^i - q_t^i) \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T} \tag{20e}$$

$$0 \leq \nu^{b}{}_t^i \perp b_t^i \geq 0,\ i \in \mathcal{S}, t \in \mathcal{T}. \tag{20f}$$

A.2 Unconstrained ISO

For the unconstrained ISO market clearing problem (8) we have that the optimality conditions are decoupled in time as well. The Lagrange function at a time instant t is given by

$$\begin{aligned}\bar{\mathcal{L}}_t(b_t^i) = \sum_{i\in\mathcal{S}} \frac{1}{2\,b_t^i}(q_t^i)^2 - \bar{p}_t \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t\right)\\ - \sum_{i\in\mathcal{S}} \underline{\nu}^{q}{}_t^i \cdot (q_t^i - \underline{q}^i) - \sum_{i\in\mathcal{S}} \overline{\nu}^{q}{}_t^i \cdot (\overline{q}^i - q_t^i),\ t \in \mathcal{T}.\end{aligned} \tag{21}$$

The optimality conditions are

$$\nabla_{q_t^i}\bar{\mathcal{L}}_t(b_t^i) = \frac{1}{b_t^i}q_t^i - p_t - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ t\in\mathcal{T} \tag{22a}$$

$$0 \le \bar{p}_t \perp \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t\right) \ge 0,\ t\in\mathcal{T} \tag{22b}$$

$$0 \le \underline{\nu}^{q}{}_t^i \perp (q_t^i - \underline{q}^i) \ge 0,\ i\in\mathcal{S}, t\in\mathcal{T} \tag{22c}$$

$$0 \le \overline{\nu}^{q}{}_t^i \perp (\overline{q}^i - q_t^i) \ge 0,\ i\in\mathcal{S}, t\in\mathcal{T}. \tag{22d}$$

A.3 Constrained ISO

For the ISO problem (11) the Lagrange function is given by

$$\begin{aligned}\mathcal{L}(b_t^i) = \;& \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{S}} \frac{1}{2\,b_t^i}(q_t^i)^2 - \sum_{t\in\mathcal{T}} p_t\left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t\right)\\ &+ \sum_{t\in\mathcal{T}^-}\sum_{i\in\mathcal{S}} \lambda_{t+1}^i(q_{t+1}^i - q_t^i - \Delta q_t^i) - \sum_{t\in\mathcal{T}^-}\sum_{i\in\mathcal{S}} \underline{\nu}^{\Delta}{}_t^i(\Delta q_t^i - \underline{r}^i) - \sum_{t\in\mathcal{T}^-}\sum_{i\in\mathcal{S}} \overline{\nu}^{\Delta}{}_t^i(\overline{r}^i - \Delta q_t^i)\\ &- \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{S}} \underline{\nu}^{q}{}_t^i(q_t^i - \underline{q}^i) - \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{S}} \overline{\nu}^{q}{}_t^i(\overline{q}^i - q_t^i) + \sum_{i\in\mathcal{S}} \lambda_k^i(q_k^i - \hat{q}_k^i).\end{aligned} \tag{23}$$

The optimality conditions are

$$\nabla_{q_T^i}\mathcal{L} = \frac{1}{b_T^i}q_T^i - p_T + \lambda_T^i - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ i\in\mathcal{S} \tag{24a}$$

$$\nabla_{q_t^i}\mathcal{L} = \frac{1}{b_t^i}q_t^i - p_t + \lambda_t^i - \lambda_{t+1}^i - \underline{\nu}^{q}{}_t^i + \overline{\nu}^{q}{}_t^i = 0,\ i\in\mathcal{S}, t\in\mathcal{T}^- \tag{24b}$$

$$\nabla_{\Delta q_t^i}\mathcal{L} = -\lambda_{t+1}^i - \underline{\nu}^{\Delta}{}_t^i + \overline{\nu}^{\Delta}{}_t^i = 0,\ i\in\mathcal{S}, t\in\mathcal{T}^- \tag{24c}$$

$$\nabla_{\lambda_{t+1}^i} = q_{t+1}^i - q_t^i - \Delta q_t^i = 0,\ i\in\mathcal{S},\ \in\mathcal{T}^-\setminus\{k\} \tag{24d}$$

$$\nabla_{\lambda_0^i} = q_0^i - \hat{q}_k^i = 0,\ i\in\mathcal{S} \tag{24e}$$

$$0 \le p_t \perp \left(\sum_{i\in\mathcal{S}} q_t^i - \sum_{j\in\mathcal{C}} d_j^t\right) \ge 0,\ t\in\mathcal{T} \tag{24f}$$

$$0 \le \overline{\nu}^{\Delta}{}_t^i \perp \overline{r}^i - \Delta q_t^i \ge 0,\ i\in\mathcal{S}, t\in\mathcal{T}^- \tag{24g}$$

$$0 \le \underline{\nu}^{\Delta}{}_t^i \perp \Delta q_t^i - \underline{r}_i \ge 0,\ i\in\mathcal{S}, t\in\mathcal{T}^- \tag{24h}$$

$$0 \le \overline{\nu}^{q}{}_t^i \perp \overline{q}^i - q_t^i \ge 0,\ i\in\mathcal{S}, t\in\mathcal{T} \tag{24i}$$

$$0 \le \underline{\nu}^{q}{}_t^i \perp q_t^i - \underline{q}^i \ge 0,\ i\in\mathcal{S}, t\in\mathcal{T}. \tag{24j}$$

A.4 Stabilizing ISO

The stabilizing ISO formulation (16) can be written as

$$
\begin{aligned}
\min_{q_t^i, \Delta q_t^i} \quad & \delta_T && (25a)\\
\text{s.t. } & \varphi_t = \sum_{i \in \mathcal{S}} \frac{1}{2\, b_t^i} (q_t^i)^2, \quad t \in \mathcal{T} && (25b)\\
& \varphi_t \cdot \eta_t = \bar{\varphi}_t, \quad t \in \mathcal{T} && (25c)\\
& \eta_t \geq \epsilon,\ t \in \mathcal{T} && (25d)\\
& \delta_{t+1} = (1 - (\eta_{t+1} - \epsilon)) \cdot \delta_t,\ t \in \mathcal{T}^- && (25e)\\
& \sum_{i \in \mathcal{S}} q_t^i \geq \sum_{j \in \mathcal{C}} d_j^t,\ t \in \mathcal{T} && (25f)\\
& q_{t+1}^i = q_t^i + \Delta q_t^i,\ i \in \mathcal{S}, t \in \mathcal{T}^- && (25g)\\
& \underline{r}^i \leq \Delta q_t^i \leq \overline{r}^i,\ i \in \mathcal{S}, t \in \mathcal{T}^- && (25h)\\
& \underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i \in \mathcal{S}, t \in \mathcal{T} && (25i)\\
& q_k^i = \hat{q}_k^i, \delta_k = \hat{\delta}_k,\ i \in \mathcal{S}. && (25j)
\end{aligned}
$$

Here, $\bar{\varphi}_t,\ t \in \mathcal{T}$ is obtained from (8). The Lagrange function is

$$
\begin{aligned}
\mathcal{L}(b_t^i) = \quad & \delta_T + \sum_{t \in \mathcal{T}} \lambda_t^{\varphi} \left(\varphi_t - \sum_{i \in \mathcal{S}} \frac{1}{2\, b_t^i} (q_t^i)^2 \right) + \sum_{t \in \mathcal{T}} \lambda_t^{\eta} (\varphi_t \eta_t - \bar{\varphi}_t) - \sum_{t \in \mathcal{T}} \nu_t^{\eta} (\eta_t - \epsilon) \\
& + \sum_{t \in \mathcal{T}^-} \lambda_{t+1}^{\delta} \left(\delta_{t+1} - (1 - (\eta_t - \epsilon)) \cdot \delta_t \right) - \sum_{t \in \mathcal{T}} p_t \left(\sum_{i \in \mathcal{S}} q_t^i - \sum_{j \in \mathcal{C}} d_j^t \right) \\
& + \sum_{t \in \mathcal{T}^-} \sum_{i \in \mathcal{S}} \lambda_{t+1}^i (q_{t+1}^i - q_t^i - \Delta q_t^i) - \sum_{t \in \mathcal{T}^-} \sum_{i \in \mathcal{S}} \underline{\nu}^{\Delta}{}_t^i (\Delta q_t^i - \underline{r}^i) - \sum_{t \in \mathcal{T}^-} \sum_{i \in \mathcal{S}} \overline{\nu}^{\Delta}{}_t^i (\overline{r}^i - \Delta q_t^i) \\
& - \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{S}} \underline{\nu}^{q}{}_t^i (q_t^i - \underline{q}^i) - \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{S}} \overline{\nu}^{q}{}_t^i (\overline{q}^i - q_t^i) + \sum_{i \in \mathcal{S}} \lambda_k^i (q_k^i - \hat{q}_k^i) + \lambda_k^{\delta} (\delta_k - \hat{\delta}_k). \qquad (26)
\end{aligned}
$$

The optimality conditions are,

$$
\begin{aligned}
&\nabla_{\delta_T}\mathcal{L} = 1 + \lambda_T^{\delta} = 0 && \text{(27a)}\\
&\nabla_{\delta_t}\mathcal{L} = \lambda_t^{\delta} - (1-(\eta_t-\epsilon))\cdot\lambda_{t+1}^{\delta} = 0,\ t\in\mathcal{T}^- && \text{(27b)}\\
&\nabla_{\eta_t}\mathcal{L} = \lambda_t^{\eta}\cdot\varphi_t - \nu_t^{\eta} + \lambda_{t+1}^{\delta}\cdot\delta_t = 0,\ t\in\mathcal{T}^- && \text{(27c)}\\
&\nabla_{\eta_T}\mathcal{L} = \lambda_T^{\eta}\cdot\varphi_T - \nu_t^{\eta} = 0, && \text{(27d)}\\
&\nabla_{\varphi_t}\mathcal{L} = \lambda_t^{\varphi} + \lambda_t^{\eta}\eta_t = 0,\ t\in\mathcal{T} && \text{(27e)}\\
&\nabla_{q_t^i}\mathcal{L} = -\lambda_t^{\varphi}\frac{1}{b_t^i}q_t^i - p_t + \lambda_t^i - \lambda_{t+1}^i + \overline{\nu}^{q}{}_t^i - \underline{\nu}^{q}{}_t^i = 0,\ i\in\mathcal{S},\ t\in\mathcal{T}^- && \text{(27f)}\\
&\nabla_{q_T^i}\mathcal{L} = -\lambda_T^{\varphi}\frac{1}{b_T^i}q_T^i - p_T + \lambda_T^i + \overline{\nu}^{q}{}_T^i - \underline{\nu}^{q}{}_T^i = 0,\ i\in\mathcal{S} && \text{(27g)}\\
&\nabla_{\Delta q_t^i}\mathcal{L} = -\lambda_{t+1}^i + \overline{\nu}^{\Delta}{}_t^i - \overline{\nu}^{\Delta}{}_t^i = 0,\ i\in\mathcal{S},\ t\in\mathcal{T}^- && \text{(27h)}\\
&\nabla_{\lambda_t^{\varphi}}\mathcal{L} = \varphi_t - \sum_{i\in\mathcal{S}}\left(\frac{1}{2\,b_t^i}q_t^i + a_t^i\right)q_t^i = 0,\ t\in\mathcal{T} && \text{(27i)}\\
&\nabla_{\lambda_t^{\eta}}\mathcal{L} = \eta_t\varphi_t - \overline{\varphi}_t = 0,\ t\in\mathcal{T} && \text{(27j)}\\
&\nabla_{\lambda_t^{\delta}}\mathcal{L} = \delta_{t+1} - (1-(\eta_t-\epsilon))\cdot\delta_t = 0,\ t\in\mathcal{T}^- && \text{(27k)}\\
&\nabla_{\lambda_{t+1}^i}\mathcal{L} = q_{t+1}^i - q_t^i - \Delta q_t^i = 0,\ i\in\mathcal{S},\ t\in\mathcal{T}^- && \text{(27l)}\\
&\nabla_{\lambda_k^i}\mathcal{L} = q_k^i - \hat{q}_k^i = 0,\ i\in\mathcal{S} && \text{(27m)}\\
&\nabla_{\lambda_k^{\delta}}\mathcal{L} = \delta_k - \hat{\delta}_k && \text{(27n)}\\
&0\le p_t\perp\left(\sum_{i\in\mathcal{S}}q_t^i - \sum_{j\in\mathcal{C}}d_j^t\right)\ge 0,\ t\in\mathcal{T} && \text{(27o)}\\
&0\le \nu_t^{\eta}\perp(\eta_t-\epsilon)\ge 0,\ t\in\mathcal{T} && \text{(27p)}\\
&0\le \overline{\nu}^{\Delta}{}_t^i\perp(\overline{r}^i - \Delta q_t^i)\ge 0,\ i\in\mathcal{S},\ t\in\mathcal{T}^- && \text{(27q)}\\
&0\le \underline{\nu}^{\Delta}{}_t^i\perp(\Delta q_t^i - \underline{r}^i)\ge 0,\ i\in\mathcal{S},\ t\in\mathcal{T}^- && \text{(27r)}\\
&0\le \overline{\nu}^{q}{}_t^i\perp(\overline{q}^i - q_t^i)\ge 0,\ i\in\mathcal{S},\ t\in\mathcal{T} && \text{(27s)}\\
&0\le \underline{\nu}^{q}{}_t^i\perp(q_t^i - \underline{q}^i)\ge 0,\ i\in\mathcal{S},\ t\in\mathcal{T}. && \text{(27t)}
\end{aligned}
$$

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