A constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

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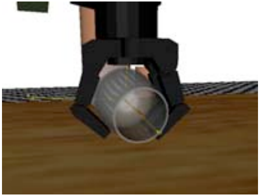
Application of Rigid Multi Body Dynamics

- RMBD in diverse areas
  - ⋆ rock dynamics
  - ⋆ human motion
  - ⋆ robotic simulations
  - ⋆ nuclear reactors
  - ⋆ virtual reality
  - ⋆ haptics
- VR or Virtual reality exposure (VRE) therapy
  - ⋆ fear of heights
  - ⋆ fear of public speaking
  - ⋆ telerehabilitation
  - ⋆ PTSD



Some Previous Approaches

- Integrate-detect-restart simulation a natural choice
  - Classical solution may not exist
  - Collisions can cause small stepsizes
- Differential algebraic equations (DAE) for joint constraints
  - Specialized techniques because non-smooth noninterpenetration and friction constraints.
- Optimization based animation technique solving a quadratic program at each step to avoid stiffness.
  - Collision detection still present, hence small stepsizes
- Penalty Barrier Methods are most popular.
  - Easy set up, even for DAEs, but problem may be stiff and requires *a priori* smoothing parameters

Hard Constraint Approaches

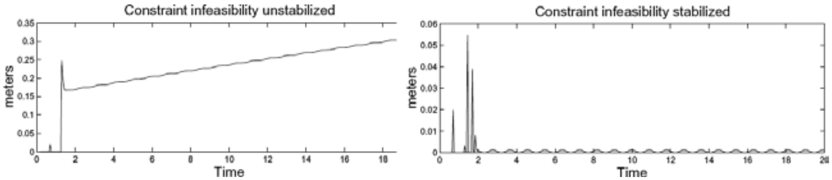
- Advantage:
  - Results are same order of magnitude as penalty method
  - Same dynamics using 4 orders of magnitude larger time step
  - We use a velocity impulse LCP based approach avoiding the lack of a solution and introducing artificial stiffness
- Disadvantage:
  - LCP model yields inequality constraints from contact and friction, treated computationally as hard constraints.

Need to Define and Compute Depth of Penetration

- To avoid infinitely small time steps, say from collisions, then minimum stepsize must exist
- For methods with minimum time step, interpenetration may be unavoidable, thus it needs to be quantified (to limit amount of interpenetration)
- Minimum Euclidean distance good for distance between objects, but not for penetration
- Note that for convex polyhedra, calculation of PD using Minkowski sums, are computationally expensive

Constraint Stabilization

- Constraint stabilization in a complementarity setting. Tackled by previous authors using
  - nonlinear complementarity problems an LCP
  - nonlinear projection (nonlinear inequality constraints)
  - post-processing method (uses potentially non-convex LCP)
  - convex LCP for constraint stabilization.
- Unlike ours, these methods need computation after solving basic LCP subproblem to achieve constraint stabilization



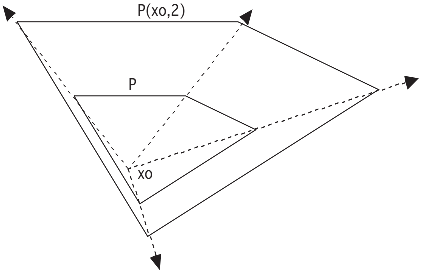
Goals

The goals of this thesis are to

- define a new computationally efficient measure that detects collision and computes penetration of two convex bodies, which is metrically equivalent to the signed Euclidean distance when close to a contact,
- develop an algorithm which efficiently models the system and solves the resulting LCP while achieving constraint stabilization, and
- implement the algorithm to simulate polyhedral multibody contact problems with friction.

A constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

- Ratio Metric
- Differentiability
- Constraints and Model
- Algorithm
- Numerical Results
- Accomplishments



Ratio Metric

- We need a new measure that defines distance and quantifies depth of penetration between convex bodies.
- We start by introducing and analyzing a new measure between two convex bodies.
- Then we extend the analysis to produce our new measure of penetration depth.
- We will see that it is metrically equivalent to the Minkowski Penetration Depth measure, but has lower computational complexity.

Polyhedra and Expansion/Contraction Maps

Definition

We define CP(A, b, x_o) to be the convex polyhedron P defined by the linear inequalities $Ax \leq b$ with an interior point x_o. We will often just write P = CP(A, b, x_o).

Definition

Let P = CP(A, b, x_o). Then for any nonnegative real number t, the expansion (contraction) of P with respect to the point x_o is defined to be

$$P(x_o, t) = \{x | Ax \leq tb + (1 - t)Ax_o\}$$

and has an associated mapping

$$\Gamma(x, x_o, t) = tx + (1 - t)x_o.$$

Minkowski Penetration Depth

Definition

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2. The Minkowski Penetration Depth (MPD) between the two bodies P_1 and P_2 is defined formally as

$$PD(P_1, P_2) = \min\{||d|| \, | interior(P_1 + d) \bigcap P_2 = \emptyset\}. \quad (1)$$

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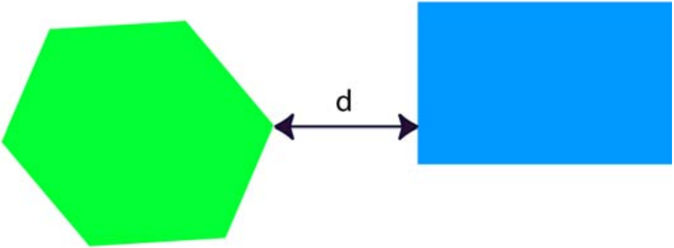

Ratio Metric Penetration Depth

Definition

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2. Then the Ratio Metric between the two sets is given by

$$r(P_1, P_2) = \min\{t | P_1(x_1, t) \bigcap P_2(x_2, t) \neq \emptyset\}, \quad (2)$$

and the corresponding Ratio Metric Penetration Depth (RPD) is given by

$$\rho(P_1, P_2, r) = \frac{r(P_1, P_2) - 1}{r(P_1, P_2)}. \quad (3)$$

Expansion/Contraction Again

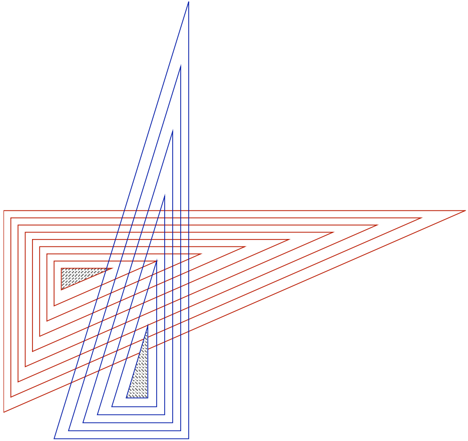

Figure: Visual representation of double expansion or contraction

Metric Equivalence Theorem

Theorem (Metric Equivalence)

Let $P_i = CP(A_i, b_i, x_i)$ be a convex polyhedron for i = 1,2, s be the MPD between the two bodies, D be the distance between x_1 and x_2, ϵ be the maximum allowable Minkowski penetration between any two bodies. Then the ratio metric penetration depth between the two sets satisfies the relationship

$$\frac{s}{D} \leq \rho(P_1, P_2, r) \leq \frac{s}{\epsilon}, \tag{4}$$

if P_1 and P_2 have disjoint interiors, and

$$-\frac{s}{\epsilon} \leq \rho(P_1, P_2, r) \leq -\frac{s}{D} \tag{5}$$

if the interiors of P_1 and P_2 are not disjoint.

Significance of the Metric Equivalence Theorem

- Let number of facets of two polyhedra be m_1 and m_2
  - Computing PD by using the Minkowski sums: $O(m_1^2 + m_2^2)$
  - Fast approximation to PD with stochastic method: $O(m_1^{3/4+\epsilon} m_2^{3/4+\epsilon})$ for any $\epsilon > 0$
  - Solving linear programming problem: $O(m_1 + m_2)$
- $\therefore$ our metric provide us with a simple way to detect collision and measure penetration of two convex polyhedral bodies bodies with lower complexity and is equivalent, for small penetration, to the classical measure
- $\therefore$ for time step h, if the MPD is $O(h^2)$ then so is the RPD

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$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(c\, u) = c\, \frac{du}{dx}$$

$$\frac{d}{dx}(u\, v) = u\, \frac{dv}{dx} + v\, \frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\, \frac{du}{dx} - u\, \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(u^n) = n\, u^{n-1} \frac{du}{dx}$$

$$\frac{d}{dx}(u \circ v) = \frac{dv}{dx}\, \left(\frac{du}{dx} \circ v\right.$$

Perfect Contact

Definition

Two convex polyhedra are in perfect contact when there is a nonempty intersection without interpenetration.

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Definition

In n-dimensional space, a Basic Contact Unit (BCU) occurs when

- two convex polyhedra are in perfect contact,
- the contact region attached to a BCU is a point, and
- exactly n+1 facets are involved at the contact.

The point where the contact occurs is called an event point, or more simply, an event.

Basic Contact Unit

- A CoF is always a BCU
- In 2D: CoF In 3D: CoF, (nonparallel) EoE
- In n-dim space, there are exactly $\left[\frac{n+1}{2}\right]$ distinct BCUs

Figure: Corner-on-Face

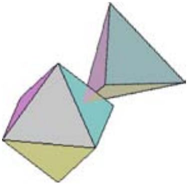

Figure: Edge-on-Edge

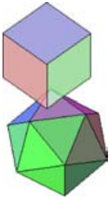

Figure: Face-on-Face

Convex Hull of BCUs

Theorem

The intersection of two convex polyhedra in perfect contact is the convex hull of the event points.

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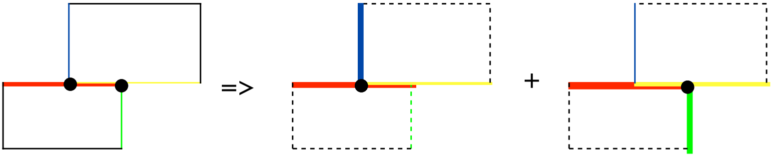

Figure: 2D Example: Contact Region Is Convex Hull of BCUs.

Nondifferentiability

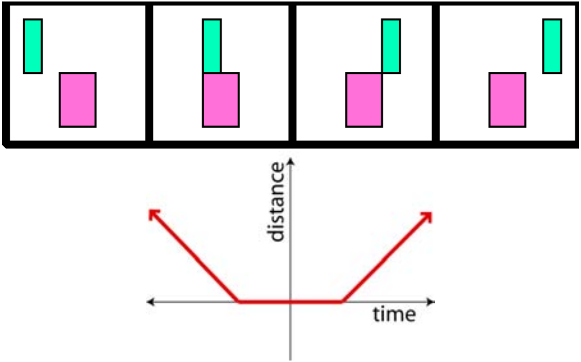

Figure: Nondifferentiability of Euclidean distance function

- In Calculus, we learn when functions are not differentiable
- Consider piecewise smooth distance function

Local/Global Coordinates

Suppose that we have $P_{L_i} = CP(A_{L_i}, b_{L_i}, 0)$ as the local representation for a convex polyhedron for i = 1, 2. The transformation from local coordinates x_{L_i} to world coordinates x is given by

$$x = x_i + R_i x_{L_i},$$

which can be rewritten into the form

$$x_{L_i} = R_i^T (x - x_i).$$

Here the matrices R_1 and R_2 are rotation matrices.

Infinite Differentiability at an Event

- If E is an event at perfect contact of convex polyhedra P_1 and P_2, then $P_E(x_i, t)$, the restrictions of $P_i(x_i, t)$ to E, is the convex body defined by the facets of $P(x_i, t)$ which involve E.
- If E is an event at perfect contact of P_1 and P_2, then

$$r(P_E(x_1, t), P_E(x_2, t)) = \min_{t \geq 0} \begin{cases} \hat{A}_{L_1} R_1^T x - \hat{b}_1 t \leq \hat{A}_{L_1} R_1^T x_1 \\ \hat{A}_{L_2} R_2^T x - \hat{b}_2 t \leq \hat{A}_{L_2} R_2^T x_2 \end{cases} \quad (6)$$

where the sum of the rows of $\hat{A}_{L_1}$ and $\hat{A}_{L_2}$ totals n+1.

- Theorem: At any event E of perfect contact, $r(P_E(x_1, t), P_E(x_2, t))$ is infinitely differentiable with respect to the translation vectors and rotation angles.

Component Functions

- Associate m^{th} event $E^{(m)}$ with component function $\widehat{\Phi}^{(m)}$
- We use the restrictions $P_{E^{(m)}}(x_1, t)$ and $P_{E^{(m)}}(x_2, t)$
- $\widehat{\Phi}^{(m)} = f(r_m)$, where $f(t) = (t-1)/t$ and

$$r_m = \min_{t \geq 0} \begin{cases} \hat{A}_{m_1} R_1^T x - b_{m_1} t \leq \hat{A}_{m_1} R_1^T x_1 \\ \hat{A}_{m_2} R_2^T x - b_{m_2} t \leq \hat{A}_{m_2} R_2^T x_2 \end{cases} \quad (7)$$

and sum of numbers of rows of $\hat{A}_{m_1}$ and $\hat{A}_{m_2}$ is n+1.

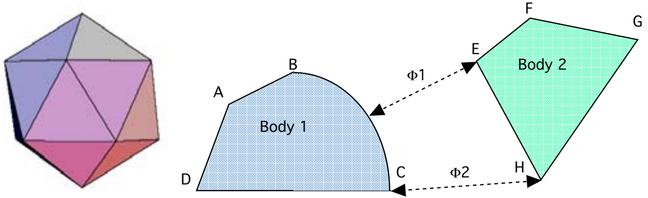

Figure: Uniqueness and Two Component Signed Distance Functions

Max of Component Functions

RPD is the maximum of component distance functions.

Theorem

Suppose $x_1 \neq x_2$ and let $P_i = CP(A_{L_i} R_i^T, b_{L_i} + A_{L_i} R_i^T x_i, x_i)$ be convex polyhedra for $i = 1, 2$ and let $\left\{ E^{(1)}, E^{(2)}, \cdots, E^{(N)} \right\}$ be the list of all possible events with corresponding component distance functions $\left\{ \widehat{\Phi}^{(1)}, \widehat{\Phi}^{(2)}, \cdots, \widehat{\Phi}^{(N)} \right\}$. Then

$$\rho(P_1, P_2, r) = \max \left\{ \widehat{\Phi}^{(1)}, \widehat{\Phi}^{(2)}, \cdots, \widehat{\Phi}^{(N)} \right\},$$

where $\rho(P_1, P_2, r)$ is defined by (3).

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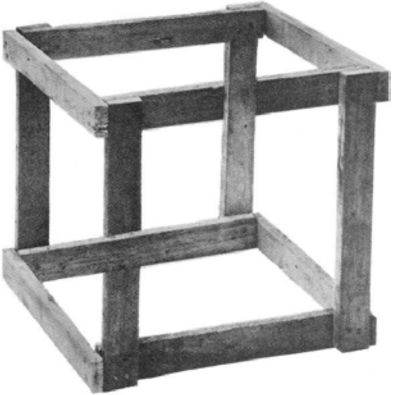

Polyhedral Bodies

- For the ${j_i}^{th}$ body, we define $P_{j_i} = CP(A_{j_j}, b_{j_i}, 0)$ to be the polyhedron defined by the linear inequalities

$$A_{j_i} x \leq b_{j_i}$$

  which contains the origin.

- Normalize this system such that all entries of vector b_{j_i} are equal to 1.

- This approach is very relevant and more robust since any body can be approximated using convex polyhedra, the prevalent representation in computer graphics.

Noninterpenetration Constraints

- Model noninterpenetration constraints by continuous piecewise differentiable signed distance functions:

$$\Phi^{(j)}(q) \geq 0, \quad j = 1, 2, \cdots, p. \tag{8}$$

- We will use RPD to compute $\Phi^{(j)}$

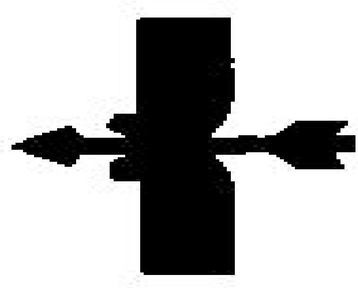

Figure: Noninterpenetration Constraint: Constraint not enforced

Joint Constraints

- Model joint constraints by sufficiently smooth $\Theta^{(i)}(q) = 0, \; i = 1, 2, \cdots, n_J$
- Define $\nu^{(i)}(q) = \nabla_q \Theta^{(i)}(q), \quad i = 1, 2, \cdots, n_J$

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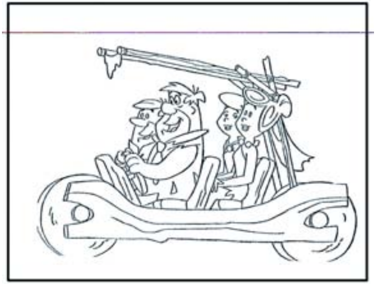

Figure: Joint Constraint: Fixed distance between wheels

Active Events $\mathcal{E}$

For two bodies in contact at position q, $\Phi^{(j)}(q) = \Phi_k^{(j)}(q) = 0$ and hence $\widehat{\Phi}^{(m)}(q) = 0$ for some m, $1 \le m \le p_o$.

Include set of imminently active events in dynamical resolution. Determine Set $\mathcal{E}$ by choosing parameters $\hat{\epsilon}_t$ and $\hat{\epsilon}_x$:

$$\begin{aligned}
\mathcal{E}_1(q) &= \left\{m \mid \Phi^{(j)} \le \hat{\epsilon}_t,\ j = Bod(E^{(m)})\right\} \\
\mathcal{E}_2(q) &= \left\{m \mid 0 \le \widehat{\Phi}^{(m)} - \Phi^{(j)} \le \hat{\epsilon}_t,\ j = Bod(E^{(m)})\right\} \\
\mathcal{E}_3(q) &= \left\{m \mid E_x^{(m)} \in CP(A_{L_{m_1}} R_{m_2}^T, b_{L_{m_1}} + A_{L_{m_1}} R_{m_1}^T x_{m_1}, x_{m_1}) + \hat{\epsilon}_x\right\} \\
\mathcal{E}_4(q) &= \left\{m \mid E_x^{(m)} \in CP(A_{L_{m_2}} R_{m_2}^T, b_{L_{m_2}} + A_{L_{m_2}} R_{m_2}^T x_{m_2}, x_{m_2}) + \hat{\epsilon}_x\right\}
\end{aligned} \tag{9}$$

and

$$\mathcal{E}(q) = \mathcal{E}_1(q) \bigcap \mathcal{E}_2(q) \bigcap \mathcal{E}_3(q) \bigcap \mathcal{E}_4(q). \tag{10}$$

ActiveEvents

- Define computationally active set (or nearly active set) by

$$\mathcal{A}(q) = \left\{ j \mid \Phi^{(j)}(q) \leq \epsilon_t, j = 1, \cdots, p \right\}, \tag{11}$$

where $\epsilon_t > 0$ is a given parameter.

- For a given position q, then $\mathcal{A}(q) = \emptyset \iff \mathcal{E}(q) = \emptyset$

Contact Model

- Normal at an event (m) : $n^{(m)}(q) = \nabla_q \widehat{\Phi}^{(m)}(q), \quad m \in \mathcal{E}$
- If one BCU per contact, complementarity of contact and compression impulse: $\widehat{\Phi}^{(m)}(q) \geq 0 \perp c_n^{(m)} \geq 0, \quad m \in \mathcal{E}$

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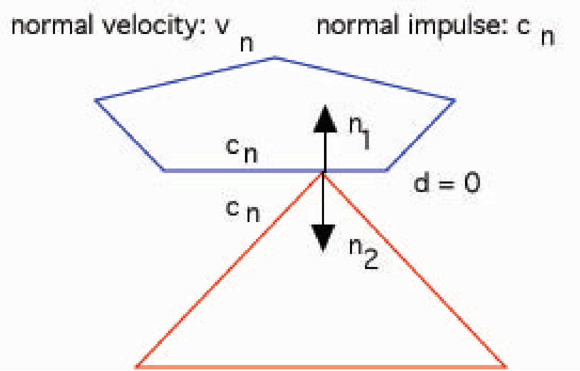

Figure: Contact Model in the case of one BCU per contact

Linear Complementarity Model

Euler discretization of the equations of motion:

$$\begin{aligned} M(q^{(l)})\left(v^{(l+1)} - v^{(l)}\right) &= h_l k\left(t^{(l)}, q^{(l)}, v^{(l)}\right) + \textstyle\sum_{i=1}^{n_J} c_\nu^{(i)} \nu^{(i)}(q^{(l)}) \\ &+ \sum_{m\in\mathcal{E}} \left(c_n^{(m)} n^{(m)}(q^{(l)}) + \sum_{i=1}^{M_C^{(m)}} \beta_i^{(m)} d_i^{(m)}(q^{(l)}) \right) \end{aligned} \tag{12}$$

Modified linearization of geometrical and noninterpenetration constraints:

$$\begin{aligned} \gamma\Theta^{(i)}(q^{(l)}) + h_l {\nu^{(i)}}^T(q^{(l)}) v^{(l+1)} &= 0, \quad i = 1, 2, \cdots, n_J, \\ {n^{(m)}}^T(q^{(l)}) v^{(l+1)} + \frac{\gamma}{h_l}\Phi^{(j)}(q^{(l)}) &\geq 0 \quad \perp c_n^{(m)} \geq 0, \qquad m \in \mathcal{E}. \end{aligned} \tag{13}$$

Friction Model

Friction model (usual classical pyramid approximation of friction cone, see Stewart & Trinkle 1995 or Anitescu & Hart 2004):

$$\begin{aligned} D^{(m)^T}(q)v + \lambda^{(m)} e^{(m)} &\geq 0 \quad \perp \quad \beta^{(m)} \geq 0, \\ \mu c_n^{(m)} - e^{(m)^T} \beta^{(m)} &\geq 0 \quad \perp \quad \lambda^{(m)} \geq 0. \end{aligned} \tag{14}$$

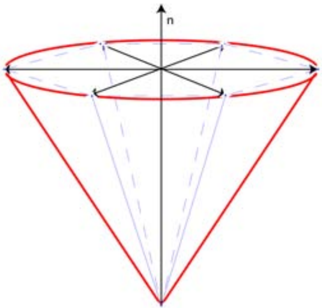

Figure: Approximation of Friction Cone

Mixed Complementarity and QP Formulation

$$
\begin{array}{llllllll}
M^{(l)} v & -\widetilde{n}\widetilde{c}_n & -\widetilde{D}\widetilde{\beta} & & = -q^{(l)} & & & \\
\widetilde{\nu}^T v & & & & = -\Upsilon & & & \\
\widetilde{n}^T v & & & -\tilde{\mu}\lambda & \geq -\Gamma - \Delta & \perp & c_n \geq 0 & \\
\widetilde{D}^T v & & & +\widetilde{E}\lambda & \geq 0 & \perp & \widetilde{\beta} \geq 0 & \\
 & \tilde{\mu} c_n & -\widetilde{E}^T \widetilde{\beta} & & \geq 0 & \perp & \lambda \geq 0 &
\end{array}
\qquad (15)
$$

Mixed Complementarity and QP Formulation

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\begin{array}{llllllll}
M^{(l)}v & -\widetilde{n}\widetilde{c}_n & -\widetilde{D}\widetilde{\beta} & & = -q^{(l)} & & & \\
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\widetilde{n}^T v & & & -\tilde{\mu}\lambda & \geq -\Gamma - \Delta & \perp & c_n \geq 0 & (15)\\
\widetilde{D}^T v & & & +\widetilde{E}\lambda & \geq 0 & \perp & \widetilde{\beta} \geq 0 & \\
 & \tilde{\mu}c_n & -\widetilde{E}^T\widetilde{\beta} & & \geq 0 & \perp & \lambda \geq 0 &
\end{array}
$$

Note (15) constitutes 1^{st}-order optimality conditions of QP

$$
\begin{array}{lrcll}
\min\limits_{v,\lambda} & \frac{1}{2}v^T M^{(l)} v + q^{(l)^T} v & & & \\
\text{s.t.} & n^{(m)^T} v - \mu^{(m)}\lambda^{(m)} & \geq & -\Gamma^{(m)} - \Delta^{(m)}, & m \in \mathcal{E} \\
 & D^{(m)^T} v + \lambda^{(m)} e^{(m)} & \geq & 0, & m \in \mathcal{E} \\
 & \nu_i^T v & = & -\Upsilon_i, & 1 \leq i \leq n_J \\
 & \lambda^{(m)} & \geq & 0 & m \in \mathcal{E}
\end{array}
\qquad (16)
$$

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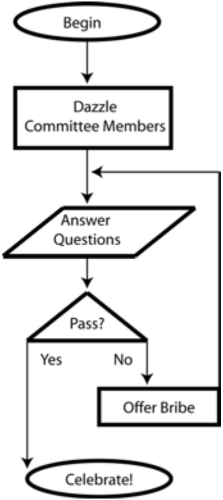

Algorithm for Nearly Active Events

Algorithm

Step 1: Solve the dual problem.

Step 2: List the active hyperplanes $H_{1i}, i = 1, \ldots, n_1$ and $H_{2j}, j = 1, \ldots, n_2$.

Step 3: Choose appropriate parameter ϵ,

Step 4a: Check H_{1i} with the list of ϵ adjacent points of H_{2j}.

Step 4b: Check H_{2j} with the list of ϵ adjacent points of H_{1i} .

Step 4c: Check ϵ adjacent edges of H_{1i} and H_{2j}.

Algorithm for Nearly Active Events

Algorithm

Step 1: Solve the dual problem.

Step 2: List the active hyperplanes $H_{1\,i}, i = 1, \ldots, n_1$ and $H_{2j}, j = 1, \ldots, n_2$.

Step 3: Choose appropriate parameter ϵ,

Step 4a: Check $H_{1\,i}$ with the list of ϵ adjacent points of H_{2j}.

Step 4b: Check H_{2j} with the list of ϵ adjacent points of $H_{1\,i}$.

Step 4c: Check ϵ adjacent edges of $H_{1\,i}$ and H_{2j}.

- Because we do not stop nor reduce time steps, we need to include events that would be active at the next step, thus we use “nearly active” events

Definition of Measure of Infeasibility

- Set of allowable positions for some $\epsilon > 0$, the sets
  - $\Omega_\epsilon^\Phi = \{q \in Q \mid \Phi^{(j)}(q) \geq -\epsilon, 1 \leq m \leq p\}$
  - $\Omega_\epsilon^\Theta = \{q \in Q \mid \left|\Theta^{(i)}(q)\right| \geq -\epsilon, i = 1, 2, \cdots, n_J\}$
  - $\Omega_\epsilon = \Omega_\epsilon^\Phi \cap \Omega_\epsilon^\Theta$
- Measure of infeasibility
  - $I(q) = \max\limits_{1 \leq j \leq p, 1 \leq i \leq n_J} \left\{\Phi_-^{(j)}(q), \left|\Theta^{(i)}(q)\right|\right\}$

Assumption A1

A1: There exists $\epsilon_o > 0$, $C_1^d > 0$, and $C_2^d > 0$ such that

- $\Phi^{(j)}$ for $1 \leq j \leq n_B$ are piecewise continuous on their domains Ω_ϵ, with piecewise components $\widehat{\Phi}^{(m)}(q)$ which are twice continuously differentiable in their respective open domains with first and second derivatives uniformly bounded by $C_1^d > 0$ and $C_2^d > 0$, respectively, and
- $\Theta^{(i)}(q)$ for $i = 1, 2, \cdots, m$ are twice continuously differentiable in Ω_ϵ with first and second derivatives uniformly bounded by $C_1^d > 0$ and $C_2^d > 0$, respectively.

Using Assumption A1

Lemma

If Assumption A1 holds, then $\Phi^{(j)}$ for $1 \leq j \leq n_B$ is everywhere directionally differentiable. Moreover, the generalized gradient of $\Phi^{(j)}$ is contained in the convex cover of the gradients of its component functions which are active at q evaluated at q.

Note: We use $\Phi^{(j)^o}(q; v) = \limsup_{p \to q, t \downarrow 0} \frac{\Phi^{(j)}(p + tv) - \Phi^{(j)}(p)}{t}$

Using Assumption A1

Lemma

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Note: We use $\Phi^{(j)^o}(q; v) = \limsup_{p \to q, t \downarrow 0} \frac{\Phi^{(j)}(p + tv) - \Phi^{(j)}(p)}{t}$

Lemma

If Assumption A1 holds, then for any j such that $1 \leq j \leq n_B$, then $\Phi^{(j)}$ satisfies a Lipschitz condition.

Note: We use Lebourg's Mean Value Theorem in the proof

Assumptions D1 - D3

D1: The mass matrix is constant. That is, $M\left(q^{(l)}\right) = M^{(l)} = M$.

D2: The norm growth parameter is constant: $c(\cdot,\cdot,\cdot) \leq c_o$

D3: The external force is continuous and increases at most linearly with the pos. and vel., and unif. bdd in time:

$$k(t,v,q) = k_o(t,v,q) + f_c(v,q) + k_1(v) + k_2(q)$$

and there is some constant $c_K \geq 0$ such that

$$\begin{aligned} \|k_o(t,v,q)\| &\leq c_K \\ \|k_1(v)\| &\leq c_K \|v\| \\ \|k_2(q)\| &\leq c_K \|q\|. \end{aligned}$$

Also assume

$$v^T f_c(v,q) = 0 \quad \forall v,q.$$

Algorithm for Piecewise Smooth RMBD

Algorithm

Algorithm for piecewise smooth multibody dynamics

Step 1: Given $q^{(l)}$. $v^{(l)}$. and h_l, calculate the active set $\mathcal{A}\left(q^{(l)}\right)$ and active events $\mathcal{E}\left(q^{(l)}\right)$.

Step 2: Compute $v^{(l+1)}$, the velocity solution of our mixed LCP .

Step 3: Compute $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$.

Step 4: IF finished, THEN stop ELSE set $l = l + 1$ and restart.

Main Result

Theorem

Consider the time-stepping algorithm defined above and applied over a finite time interval [0, T]. Assume that

- *The active set* $\mathcal{A}(q)$ *is defined by (11)*
- *The active events* $\mathcal{E}(q)$ *are defined by (10)*
- *The time steps* $h_l > 0$ *satisfy*
$$\sum_{l=0}^{N-1} h_l = T \quad \text{and} \quad \frac{h_{l-1}}{h_l} = c_h, \quad l = 1, 2, \cdots, N-1$$
- *The system satisfies Assumptions (A1) and (D1) - (D3)*
- *The system is initially feasible. That is,* $I(q^{(0)}) = 0$

Then, there exist $H > 0$, $V > 0$, *and* $C_c > 0$ *such that*

$$\left|\left|v^{(l)}\right|\right| \leq V \quad \text{and} \quad I\left(q(l)\right) \leq C_c \left|\left|v^{(l)}\right|\right|^2 h_{l-1}^2, \quad \forall l,\ 1 \leq l \leq N$$

From of Proof

- Proof proceeds similarly to proof in Anitescu & Hart 2004 and used a Theorem in the same paper
- We use Lebourg's Mean Value Theorem which states that given q_1 and q_2 in the domain of $\Phi^{(j)}$, there exists q_o on the line segment between q_1 and q_2 that satisfies

$$\Phi^{(j)}(q_1) - \Phi^{(j)}(q_2) \in \left\langle \partial\Phi^{(j)}(q_o), q_1 - q_2 \right\rangle .$$

This means that there is some $\Gamma \in \partial\Phi^{(j)}$ such that

$$\Phi^{(j)}(q_1) - \Phi^{(j)}(q_2) = \Gamma(q_1 - q_2).$$

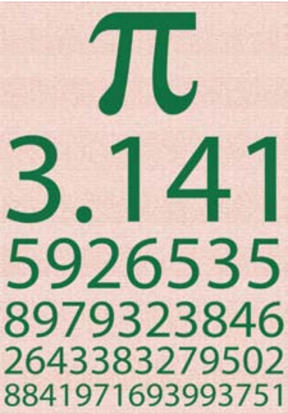
Here $\partial\Phi^{(j)}$ is the generalized gradient.

Consequences of the Theorem

- Algorithm achieves constraint stabilization because the infeasibility is bounded above by the size of the solution. In particular, $v^{(l+1)} = 0 \Rightarrow I(q^{(l+1)}) = 0$
- Linear $O(h)$ method yields quadratic $O(h^2)$ infeasibility
- Velocity remains bounded
- No need to change the step size to control infeasibility
- Solve one linear complementarity problem per step

A constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

- Ratio Metric
- Differentiability
- Constraints and Model
- Algorithm
- Numerical Results
- Accomplishments

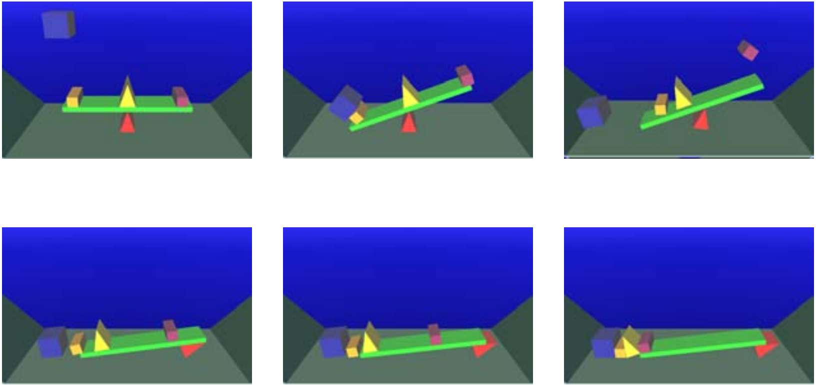


Explanation of Parameters

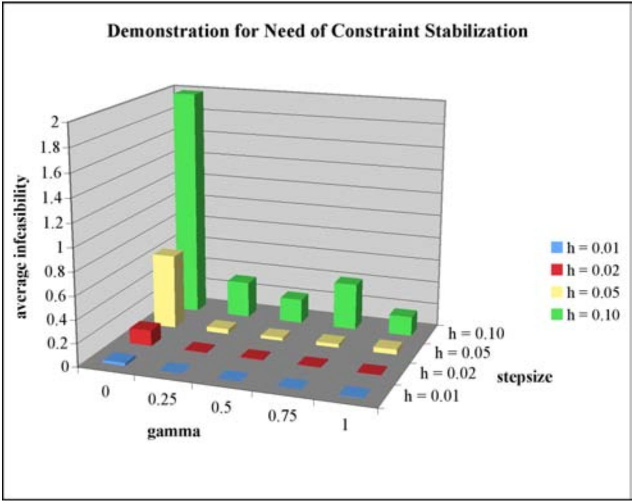
We successfully implement our algorithm for numerous examples, and in all simulations, we define the following parameters:

h	is the constant stepsize,
μ	is the Coulomb friction coefficient,
γ	is the constraint stabilization parameter.
ϵ_x	is an event detection parameter,
ϵ_t	is an event detection parameter,
ϵ_0	is an event detection parameter, and
δ_{max}	is the maximum allowable determinant.

Six successive frames from Balance2

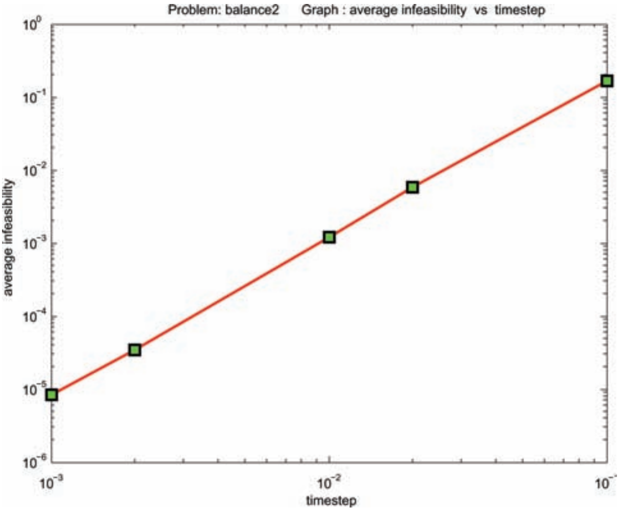


Balance2



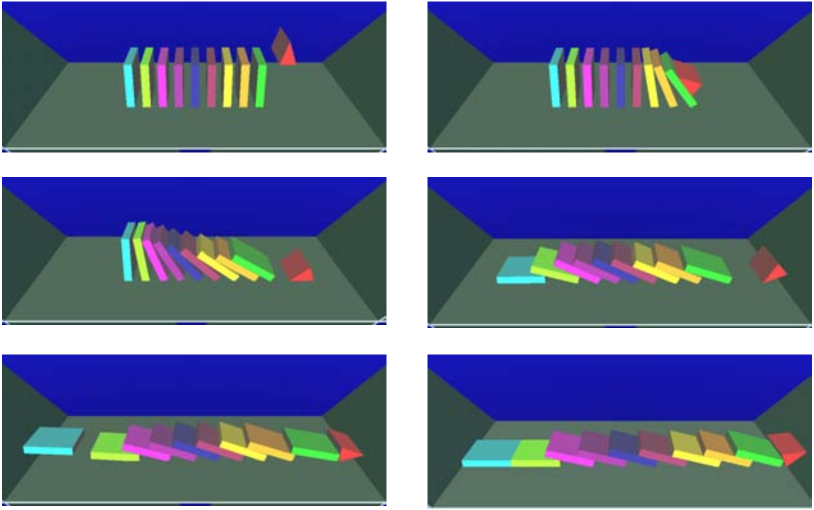
Smaller stepsize ⇒ smaller average infeasibility
Constraint stabilization ⇒ smaller average infeasibility

Balance2

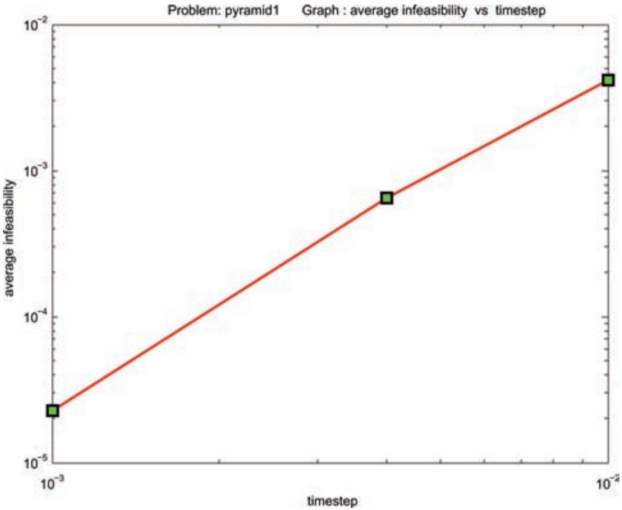


Average infeasibility shows quadratic $O(h^2)$ nature

Six successive frames from Pyramid1

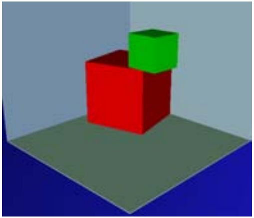
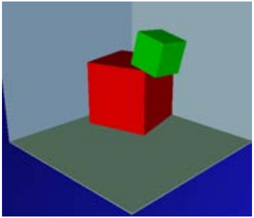
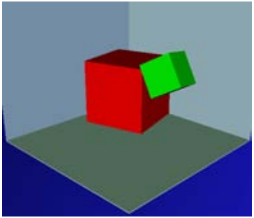
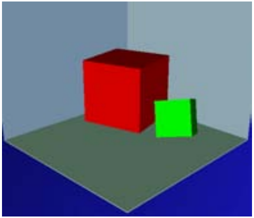


Pyramid1

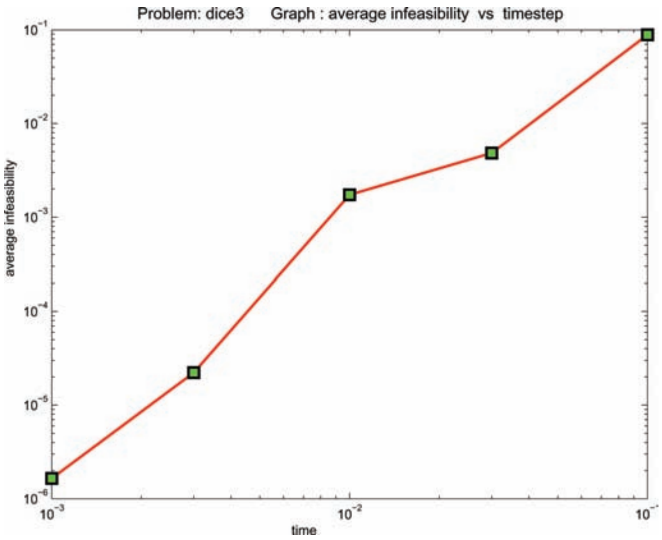


Quadratic convergence of average infeasibility

Four successive frames from Dice3

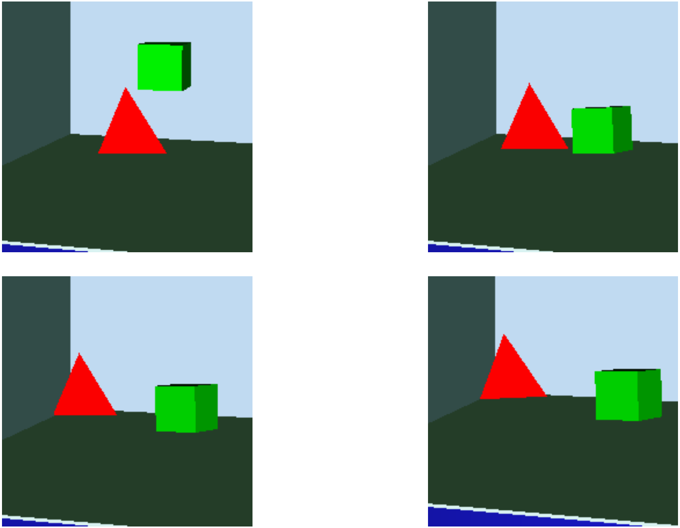





Dice3

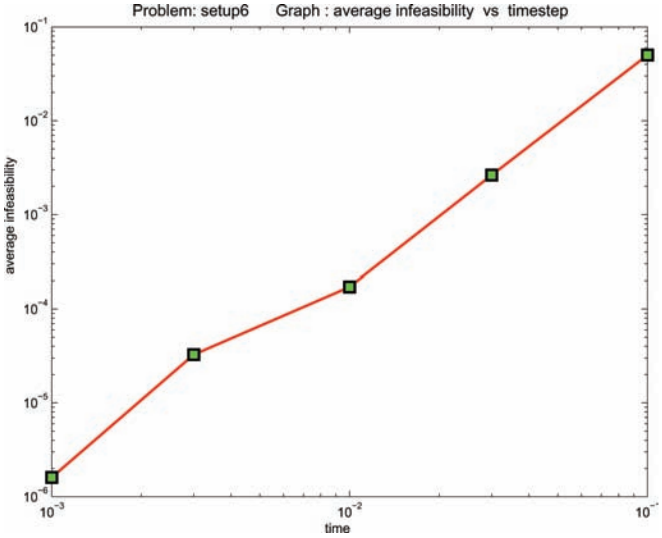


Average infeasibility demonstrates $O(h^2)$ nature

Four successive frames from Setup6



Setup6



Once again, an indication of $O(h^2)$ convergence

A constraint-stabilized time-stepping approach for piecewise smooth multibody dynamics

- Ratio Metric
- Differentiability
- Constraints and Model
- Algorithm
- Numerical Results
- Accomplishments

Accomplishments from This Thesis

- Successfully developed a computationally efficient signed distance function, Ratio Metric
- Successfully shown equivalence of RPM to MPD
- Successfully calculated generalized gradients and showed that infeasibility at step l is upper bounded by $O(||h_{l-1}||^2 \, ||v^{(l)}||^2)$
- Successfully developed and analyzed algorithm that achieves constraint stabilization solving one LCP per step
- Successfully implemented this algorithm for several problems with good results

List of Publications

- M. Anitescu and G. D. Hart, Solving nonconvex problems of multibody dynamics with joints, contact, and small friction by successive convex relaxation, Mechanics Based Design of Structures and Machines, 31 (2003), pp. 335-356.
- M. Anitescu and G. D. Hart,, A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction, International Journal for Numerical Methods in Engineering, 60 (2004), pp. 2335-2371.
- M. Anitescu and G. D. Hart, A fixed-point iteration approach for multibody dynamics with contact and small friction, Mathematical Programming, 101 (2004), pp. 3-32.
- M. Anitescu, A. Miller, and G. D. Hart, Constraint stabilization for time-stepping approaches for rigid multibody dynamics with joints, contact and friction, in Proceedings of the 2003 ASME International Design Engineering Technical Conferences, Chicago, Illinois, 2003, American Society for Mechanical Engineering. ANL/MCS-P1023-0403.
- G. D. Hart and M. Anitescu, A hard-constraint time-stepping approach for rigid multibody dynamics with joints, contact, and friction, in Proceedings of the Richard Tapia Celebration of Diversity in Computing Conference 2003, J. Meza and B. York, eds., New York, NY, USA, 2003, ACM Press, pp. 34-41.
- Publications in preparation: One dealing with Depth of Penetration by Linear Programming, the other dealing with Constraint Stabilization for Nonsmooth Shapes.

Future Research

- I plan to demonstrate that computation of RPD is faster than computation of MPD
- I plan to optimize the algorithm. For example, I need to find a rigorous way to reduce the number of active gradients
- I plan to evaluate the bounds of constraint stabilization, because it would be interesting to explore the possibility of constraint stabilization results being useful for values of $\gamma \geq 1$
- I plan to increase the library of successfully solved examples, including the famous Brazil Nut problem

Thank You! Thank You! Thank You! Thank You! Thank You!

- Dr. Mihai Anitescu, Department of Mathematics

Thank You! Thank You! Thank You! Thank You! Thank You!

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- Dr. William J. Layton, Department of Mathematics

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- Dr. Beatrice M. Riviere, Department of Mathematics
- Dr. Andrew J. Schaefer, Department of Ind. Engineering
- Dr. Ivan P. Yotov, Department of Mathematics

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